Final Exam
Math 340 Section 202 April 12th 2012

Name ______________________________ **Student Number** ______________________________

Signature ____

The exam is 150 minutes long and worth a total of 100 points. No books, notes or calculators may be used. **Show all of your work, simplify your answers, and justify your answers carefully**. You will be graded on the clarity of your exposition as well as the correctness of your answers.

Good luck.

UBC Rules governing examinations:

(a) Each candidate should be prepared to produce his/her UBCcard upon request for identification.

(b) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.

(c) No candidate shall be permitted to enter the examination room after the expiration of one half hour from the scheduled starting time, or to leave during the first half hour, *or the last 15 minutes* of the examination.

(d) Candidates guilty of any of the following or similar dishonest practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

a) Making use of any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including phones), or other memory aid devices other than those authorized by the examiners.

b) Speaking or communicating with other candidates.

c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness will not be received.

(e) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

Useful Formulae Page

The following formulae may be of use. You are assumed to understand what they mean.

$$\begin{aligned}\mathbf{x}_B &= B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N \\ z &= \mathbf{c}_B B^{-1}\mathbf{b} + (\mathbf{c}_N - \mathbf{c}_B B^{-1}A_N)\mathbf{x}_N\end{aligned}$$

and $\mathbf{y}B = \mathbf{c}_B,\ B\mathbf{d} = \mathbf{a},\ \mathbf{x}_B^* - t\mathbf{d}$.

1 (10 points)**.** *Linear programs:* You and your friends are studying the following problem.

$$\begin{array}{lrrrcr} \text{maximize} & 5x_1 & -2x_2 & -3x_3 & & \\ \text{subject to} & x_1 & +x_2 & & \leq & 2 \\ & 2x_1 & +x_2 & -x_3 & \leq & 2 \\ & 3x_1 & -x_2 & -2x_3 & \leq & -3 \\ & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

(a) Your friend Alice claims the problem is infeasible. Without using the simplex method explain why she is incorrect.

(b) Your friend Bob claims the problem is unbounded. Without using the simplex method explain why he is incorrect, by showing that the optimal value can be no greater than -1.

(c) The following dictionary has been produced by solving a certain LP problem in standard form:

$$\begin{array}{rcrcrcrcr} s_1 & = & 3 & - & x_1 & - & x_3 & + & s_3 \\ x_2 & = & 12 & - & x_1 & - & 2x_3 & - & s_3 \\ s_2 & = & 4 & - & x_1 & + & x_3 & + & s_3 \\ z & = & 24 & - & x_1 & - & 3x_3 & - & 2s_3 \end{array}$$

Find the original LP problem in standard form.

2 (15 points). *Duality:*

(a) (5 points) State the Weak Duality Theorem.

(b) (10 points) Consider a primal LP problem with a feasible solution x^* that yields z^* when substituted into the objective function z. Let its dual LP problem have a feasible solution y^* that yields w^* when substituted into its objective function w.

Prove that if $z^* = w^* = \alpha$ then α is the optimal value for *both* problems.

3 (15 points). *Complementary slackness:* Archaeologists find the following fragment in 3012.

$$\begin{array}{lrcrcrcrcr} \textit{maximize} & 3x_1 & + & 2x_2 & + & 6x_3 & + & x_4 & & \\ \textit{subject to} & 2x_1 & + & x_2 & + & 4x_3 & + & 2x_4 & \leq & 24 \\ & 3x_1 & + & x_2 & + & x_3 & + & 4x_4 & \leq & 15 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

has optimal value ? *with optimal solution* $x_2 =?$, $x_3 =?$ *and* $x_1 = x_4 = 0$.

Use complementary slackness to fill in the three ?
– that is, find the optimal value, x_2 and x_3.

4 (20 points). *Pivots and dual pivots:* This question has three parts, and the dictionary is repeated overleaf for your convenience.

Consider the LP problem

$$\begin{array}{rrcrcl} \text{maximize} & x_1 & + & (2-a)x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & b \\ & x_1 & + & 2x_2 & \leq & 2 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

where a and b are real numbers.

Introducing slack variables s_1 and s_2, here is one possible dictionary during the simplex method:

$$\begin{array}{rcccccc} x_2 & = & b & - & x_1 & - & s_1 \\ s_2 & = & 2-2b & + & x_1 & + & 2s_1 \\ z & = & 2b-ab & + & (a-1)x_1 & + & (a-2)s_1 \end{array}$$

(a) For what range of a, b is the above dictionary optimal? Explain your answer.

(b)

$$\begin{array}{rcrcrcr} x_2 & = & b & - & x_1 & - & s_1 \\ s_2 & = & 2-2b & + & x_1 & + & 2s_1 \\ z & = & 2b-ab & + & (a-1)x_1 & + & (a-2)s_1 \end{array}$$

When $b = 1/2$ and $a = 3/2$, perform one regular simplex pivot to obtain an optimal dictionary. If you perform this same pivot with the variables a and b not set to certain values then for what range of a, b is this new dictionary optimal?

(c)

$$
\begin{array}{rclclcl}
x_2 & = & b & - & x_1 & - & s_1 \\
s_2 & = & 2-2b & + & x_1 & + & 2s_1 \\
z & = & 2b-ab & + & (a-1)x_1 & + & (a-2)s_1
\end{array}
$$

When $b = 3/2$ and $a = 1/2$, perform one *dual* pivot to obtain an optimal dictionary. If you perform this same pivot with the variables a and b not set to certain values then for what range of a, b is this new dictionary optimal?

5 (20 points). *Revised simplex method:*

$$\begin{array}{lrrrcr} \text{Maximize} & 2x_1 & +3x_2 & +3x_3 & & \\ \text{subject to} & 3x_1 & +x_2 & & \leq & 40 \\ & -x_1 & +x_2 & +4x_3 & \leq & 20 \\ & 2x_1 & -2x_2 & +5x_3 & \leq & 15 \\ & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

Solve this problem using the *revised simplex method* and eta factorization. Use the largest coefficient rule to select your entering and leaving variables. (You should find you are stopped during the third iteration.)

6 (20 points). *Sensitivity analysis:* We run an ice cream stall downtown with three luxury flavours: Voluptuous Vanilla, Capricious Chocolate and Raspberry Ripple. To make a vat of Voluptuous Vanilla we require 20 kg milk, 20 kg egg yolks and 10 kg sugar. To make a vat of Capricious Chocolate we require 10 kg milk, 10 kg egg yolks and 10 kg sugar. To make a vat of Raspberry Ripple we require 30 kg milk, 20 kg egg yolks and 30 kg sugar. On each vat of Voluptuous Vanilla we make \$ 20 profit, each vat of Capricious Chocolate \$ 20 profit, each vat of Raspberry Ripple \$ 10 profit.

We can afford 180 kg milk, 240 kg egg yolks and 150 kg sugar per day. If x_1, x_2, x_3 are the vats we sell respectively of Voluptuous Vanilla, Capricious Chocolate and Raspberry Ripple per day we get the following Linear Programming problem.

$$\text{maximise } 2x_1 + 2x_2 + x_3 \text{ subject to } x_1, x_2, x_3 \geq 0 \text{ and}$$

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\leq 18 \\ 2x_1 + x_2 + 2x_3 &\leq 24 \\ x_1 + x_2 + 3x_3 &\leq 15 \end{aligned}$$

After applying the simplex method we get the final dictionary

$$\begin{array}{rclllll} x_1 &=& 3 & & -s_1 & +s_3 \\ x_2 &=& 12 & -3x_3 & +s_1 & -2s_3 \\ s_2 &=& 6 & +x_3 & +s_1 & \\ z &=& 30 & -5x_3 & & -2s_3 \end{array}$$

having used at this step

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{pmatrix}$$

(a) (5 points) A vat of Slinky Strawberry can be made from 10 kg milk, 20 kg egg yolks and 10 kg sugar at a profit of \$ 25. Is it profitable to produce it?

(b) (5 points) If the objective function is changed to $(2-\gamma)x_1 + 2x_2 + (1+\gamma)x_3$ determine the range on γ so that the basis $\{x_1, x_2, s_2\}$ remains optimal.

(c) (5 points) Returning to the original problem: if we can now afford 200 kg of sugar per day what is the new optimal value, and optimal solution?

(d) (5 points) Returning to the original problem: Our ice cream machine ensures that we make 2 vats of Voluptuous Vanilla for every 1 vat of Capricious Chocolate, and we can't make more than a combined total of 120 kg a day of these two flavours. Add this new constraint to the final dictionary given and hence find a new optimal solution.