Final Exam
Math 340 Section 202 April 20th 2011

Name ______________________________**Student Number** ______________________________

Signature ____

The exam is 150 minutes long and worth a total of 100 points. No books, notes or calculators may be used. **Show all of your work, simplify your answers, and justify your answers carefully**. You will be graded on the clarity of your exposition as well as the correctness of your answers.

Good luck.

UBC Rules governing examinations:

(a) Each candidate should be prepared to produce his/her UBCcard upon request for identification.

(b) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.

(c) No candidate shall be permitted to enter the examination room after the expiration of one half hour from the scheduled starting time, or to leave during the first half hour, *or the last 15 minutes* of the examination.

(d) Candidates guilty of any of the following or similar dishonest practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

a) Making use of any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including phones), or other memory aid devices other than those authorized by the examiners.

b) Speaking or communicating with other candidates.

c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness will not be received.

(e) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

Useful Formulae Page

The following formulae may be of use. You are assumed to understand what they mean.

$$\begin{aligned} \mathbf{x}_B &= B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N \\ z &= \mathbf{c}_B B^{-1}\mathbf{b} + (\mathbf{c}_N - \mathbf{c}_B B^{-1}A_N)\mathbf{x}_N \end{aligned}$$

and $\mathbf{y}B = \mathbf{c}_B,\ B\mathbf{d} = \mathbf{a},\ \mathbf{x}_B^* - t\mathbf{d}$.

1 (10 points)**.** *Unbounded LP problems:* Using proof by contradiction, show that the following LP problem is unbounded.

$$
\begin{array}{lrcrcrcr}
\text{Maximize} & 3x_1 & + & 4x_2 & + & 7x_3 & & \\
\text{subject to} & 2x_1 & + & 3x_2 & - & x_3 & \leq & -3 \\
 & x_1 & - & x_2 & & & \leq & 4 \\
 & -3x_1 & + & x_2 & - & 2x_3 & \leq & 2 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

2 (15 points)**.** *Duality:*

(a) (5 points) State the Strong Duality Theorem.

(b) (10 points) For each of the following statements state if it is *true* or *false*. If you state true, then give a proof of the statement. If you state false then give a counter example by stating an LP problem, its dual and corresponding optimal solution(s).

i) If a primal LP problem has an optimal solution then the dual LP problem always has an optimal solution.

ii) If a primal LP problem has a **unique** optimal solution then the dual LP problem always has a **unique** optimal solution.

3 (20 points). *Complementary slackness:*

NOTE: **There are four parts to this questions in total.**

Golem enters a cave that contains 4 sacks of different precious powders. In the cave there are a_i kilos of powder number i. A sack full of powder i is worth c_i silver pennies and weighs a_i kilos. Golem can carry up to b kilos out of the cave.

Let x_i denote the the weight in kilos of powder i which Golem carries.

You have the following information:

$$a_1 = 2\ ,\ a_2 = 3\ ,\ a_3 = 2\ ,\ a_4 = 4,$$

$$c_1 = 600\ ,\ c_2 = 600\ ,\ c_3 = 200\ ,\ c_4 = 20,$$

and $b = 6$.

(a) (4 points) Formulate Golem's problem of maximizing the value of the inventory of powders he carries out of the cave as an LP problem in standard form in decision variables x_1, x_2, x_3, x_4.

(b) (4 points) Formulate the dual of this LP problem in decision variables y_1, y_2, y_3, y_4, y_5, and write down the units of the variables.

(c) (5 points) Use the first two parts of the question and complementary slackness to verify that Golem can maximize his wealth by carrying:

- all of powder 1
- all of powder 2
- 1 kilo of powder 3
- and none of powder 4.

(d) (7 points) For this part of the question, the values of the a_i's c_i's and b are not known, but they satisfy the following inequalities:

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \frac{c_3}{a_3} \geq \frac{c_4}{a_4},$$

$a_1 + a_2 < b$, and $a_1 + a_2 + a_3 > b$.

Use the first two parts of the question and complementary slackness to verify that Golem can maximize his wealth by carrying:

- all of powder 1
- all of powder 2
- $b - a_1 - a_2$ kilos of powder 3
- and none of powder 4.

4 (15 points)**.** *Linear algebra of LP problems:* Recall the revised simplex formulae from the Useful Formulae Page:

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N$$

$$z = \mathbf{c}_B B^{-1}\mathbf{b} + \left(\mathbf{c}_N - \mathbf{c_B}B^{-1}A_N\right)\mathbf{x}_N$$

Suppose at some stage of a revised simplex method for an LP problem you have a feasible basis with

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{x}_N = \begin{pmatrix} x_4 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \quad A_N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{x}_B^* = B^{-1}\mathbf{b} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix},$$

as well as $\mathbf{c}_B B^{-1}\mathbf{b} = 0$,

$$\mathbf{c}_N - \mathbf{c}_B B^{-1}A_N = \begin{pmatrix} 1 & -1 & -1 & -1 \end{pmatrix},$$

and

$$\mathbf{d} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$$

satisfies $B\mathbf{d} = \mathbf{a}$ where $\mathbf{a} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$.

Without calculating B:

(a) (6 points) Show that corresponding LP problem is unbounded.

(b) (9 points) Give a feasible solution to the corresponding LP problem such that $z \geq 1000$.

5 (20 points). *Revised simplex method:* Solve this problem using the *revised simplex method*. Use the largest coefficient rule to select your entering and leaving variables. (You should find you are stopped during the third iteration.)

$$\begin{array}{lrrrrcl} \text{Maximize} & 2x_1 & +7x_2 & +3x_3 & +x_4 & & \\ \text{subject to} & x_1 & +2x_2 & -3x_3 & +x_4 & \leq & 5 \\ & 2x_1 & & +x_3 & +3x_4 & \leq & 2 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

6 (20 points). *Sensitivity analysis:* We run a designer shoe company with three lines: Wacky Wedges, Posh Platforms, and Sharp Stilettos. To make a crate of Wacky Wedges we require 1 kg leather, 2 kg glue and 2 kg wood. To make a crate of Posh Platforms we require 2 kg leather, 1 kg glue and 1 kg wood. To make a crate of Sharp Stilettos we require 2 kg leather, 2 kg glue and 1 kg wood. On each crate of Wacky Wedges we make \$ 2 K profit, each crate of Posh Platforms \$ 3 K profit, each crate of Sharp Stilettos \$ 2 K profit.

We can afford 12 kg leather, 15 kg glue and 16 kg wood per day. If x_1, x_2, x_3 are the crates we sell respectively of Wedges, Platforms and Stilettos per day we get the following Linear Programming problem.

$$\text{maximise } 2x_1 + 3x_2 + 2x_3 \text{ subject to } x_1, x_2, x_3 \geq 0 \text{ and}$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &\leq 12 \\ 2x_1 + x_2 + 2x_3 &\leq 15 \\ 2x_1 + x_2 + x_3 &\leq 16 \end{aligned}$$

After applying the simplex method we get the final dictionary

$$\begin{array}{rcrrrr} x_1 &=& 6 & -\frac{2}{3}x_3 & +\frac{1}{3}s_1 & -\frac{2}{3}s_2 \\ x_2 &=& 3 & -\frac{2}{3}x_3 & -\frac{2}{3}s_1 & +\frac{1}{3}s_2 \\ s_3 &=& 1 & +x_3 & & +s_2 \\ z &=& 21 & -\frac{4}{3}x_3 & -\frac{4}{3}s_1 & -\frac{1}{3}s_2 \end{array}$$

having used at this step

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

(a) (5 points) Which resource has the greatest profit per resource? How much is it? Explain your answer briefly.

(b) (5 points) A colleague says that increasing the amount of wood we can afford per day will not increase our profit. Verify they are correct.

(c) (5 points) If the objective function is changed to $(2+2\gamma)x_1 + (3-\gamma)x_2 + 2x_3$ determine the range on γ so that the basis $\{x_1, x_2, s_3\}$ remains optimal.

(d) (5 points) Returning to the original problem: If we decide to put ribbons on our *Wedges and Stilettos*, and each crate of shoes requires 1 kg of ribbon, and we can afford 5 kg of ribbon per day, what is the new optimal value and optimal solution?