This examination has 12 pages including this cover

The University of British Columbia

Final Examination – 11 Dec 2009

Mathematics 340

Linear Programming

Closed book examination

Time: 150 minutes

Name____________________________ **Signature** ____________________________

UBC Student Number__________________

Special Instructions:

- Students are invited to write on both sides of each sheet.
- To receive full credit, all answers must be supported with clear and correct derivations.
- No calculators, notes, or other aids are allowed.

1		12
2		12
3		12
4		12
5		6
6		15
7		16
8		15
Total		100

[12] **1.** Consider the following problem:

$$\begin{array}{ll} \text{maximize} & \zeta = -5x_1 + 6x_2 - 4x_3 \\ \text{subject to} & \quad 2x_1 + 3x_2 - \;\; x_3 \le -2 \\ & -x_1 + 2x_2 + 2x_3 \le \;\; 3 \\ & \quad x_1, x_2, x_3 \ge 0 \end{array}$$

(a) Write the dual problem.

(b) Show that the dual problem is unbounded, by presenting a sequence of feasible inputs for the dual problem whose objective values diverge to $-\infty$.
(*One reasonable approach starts with a sketch.*)

(c) What does the result in part (b) tell us about the problem stated above?

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[12] **2.** Consider the following problem.

$$\begin{array}{llrrrrl}
\text{maximize} & c_1x_1 + c_2x_2 + c_3x_3 & & & & & \\
\text{subject to} & 3x_1 + & x_2 - & x_3 & \leq & -2 & \\
 & x_1 - & x_2 - & 2x_3 & \leq & -3 & \qquad (*) \\
 & x_1 & & & \leq & 2 & \\
 & x_1, x_2, x_3 \geq 0 & & & & &
\end{array}$$

(a) Solve problem $(*)$ when $(c_1, c_2, c_3) = (-3, -4, -2)$.

(b) Solve problem $(*)$ when $(c_1, c_2, c_3) = (7, 0, -2)$.

(c) Suppose $(c_1, c_2, c_3) = (7, k, -2)$ in problem $(*)$. Find a value of k for which problem $(*)$ has more than one maximizing point. For this k, display two different maximizers.

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[12] **3.** The matrix below shows the rewards to the column player in a standard zero-sum matrix game.

$$G = \begin{bmatrix} -1 & 4 & 2 \\ 7 & -1 & 1 \end{bmatrix}$$

(a) Set up a linear program to find the optimal mixed strategy for the column player.

(b) Solve the LP in part (a). (*Do at most four pivots.*)

(c) Find the optimal mixed strategy for the row player.

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[12] **4.** In the process of solving "max $\left\{c^T x \ : \ Ax = b, \ x \geq 0\right\}$" by the Revised Simplex Method (RSM), where

$$A = \begin{bmatrix} 3 & 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & -1 & -2 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix},$$

$$c^T = \begin{bmatrix} 2 & 1 & -2 & -1 & 0 & 0 & 0 \end{bmatrix}$$

we encounter the current basis $\mathcal{B} = \{3, 4, 7\}$. **Use the RSM** to complete the following tasks.

(a) Find the current Basic Feasible Solution x, and its objective value.

(b) Find the next entering variable (if there is one).

(c) Find the next leaving variable (if there is one).

(d) Find the new basic feasible solution after one pivot.

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[6] **5.** This problem involves an LP where some cost coefficients are not specified:

$$
\begin{array}{llcl}
\text{maximize } \zeta = c_1x_1 + c_2x_2 & & & \\
\text{subject to} & x_1 + x_2 & & \leq 4 \\
& \phantom{x_1 +{}} x_2 + x_3 & & \leq 2 \\
& x_1 \phantom{{}+ x_2} - x_3 & & \leq 3 \\
& x_1, x_2, x_3 \geq 0 & &
\end{array}
$$

Suppose a dual minimizer has the form $y^* = (3, 1, \gamma)$ for some γ. Find the maximum value of ζ.

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[15] **6.** Rowena plays the rows and Callum plays the columns in a standard zero-sum matrix game in which the rewards to Callum are displayed in the following matrix:

$$G = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 1 \\ -1 & 4 \\ -2 & 5 \end{bmatrix}.$$

(a) Write a short, clear definition of "Nash equilibrium" applicable to zero-sum games. Use only words: no mathematical symbols or variables are allowed.

(b) Consider the mixed strategies $\widetilde{\mathbf{x}} = \left(\frac{1}{4}, \frac{3}{4}\right)$ for Callum and $\mathbf{y}^* = \left(0, 0, \frac{5}{6}, \frac{1}{6}, 0\right)$ for Rowena. Are these strategies in Nash equilibrium? Explain, making reference to your definition in part (a).

(c) Find all strategies $\mathbf{x}$ for Callum (if any) that can participate in a Nash equilibrium with Rowena's choice of $\mathbf{y}^*$ from (b).

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[16] **7.** The linear program shown on the left below involves a scalar parameter k. When $k = 5$, the corresponding optimal dictionary is shown on the right:

$$\begin{array}{ll} \text{maximize } \zeta = 4x_1 + 5x_2 & \\ \text{subject to} & x_1 + x_2 \leq k \\ & x_1 + 2x_2 \leq 8 \\ & 2x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{array}
\qquad\Bigg|\qquad
\begin{array}{rl} \zeta &= 23 - 3x_3 - x_4 \\ \hline x_1 &= 2 - 2x_3 + x_4 \\ x_2 &= 3 + x_3 - x_4 \\ x_5 &= 1 + 3x_3 - x_4 \end{array}$$

(We have used $(x_3, x_4, x_5) = (w_1, w_2, w_3)$ as the slack variables for the constraints, in order.)

(a) Write a 3×5 matrix A and column vector $\mathbf{b}$ for which the constraints in the original problem can be expressed as $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Then, in the notation of the Revised Simplex Method, find the matrices B and N associated with the optimal dictionary.

(b) Use the optimal dictionary above to find B^{-1}. Explain your method.
(*Most marks here are for a clear, RSM-based explanation/derivation. Simply finding B^{-1} by some unrelated method will not earn much.*)

(c) The given dictionary is correct only when $k = 5$. Find the general form of this dictionary in terms of k; of course, substituting $k = 5$ should restore the dictionary above.

(d) For each $k \geq 0$, find the set of maximizers for the stated problem. Also calculate and sketch the graph of the maximum value, $\zeta_{\max}(k)$, as a function of $k \geq 0$.

(e) Identify a specific value of $k > 0$ for which the dual problem has more than one minimizer. Find all dual minimizers for the k you choose.

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[15] **8.** Our factory makes portable computer memory sticks in two styles. The Hello Kitty model, aimed at Facebook lovers, requires one logic circuit, one crypto chip, and two memory chips; each assembled unit earns a profit of 4 dollars. The Black Lightning model, styled for paranoid hackers, requires one logic circuit, two crypto chips, and one memory chip; each unit earns a profit of 5 dollars. We just got a new shipment of parts (the factory was empty!): 5000 logic circuits, 8000 crypto chips, and 8000 memory chips. Suppose we manufacture $1000x_1$ Hello Kitty units and $1000x_2$ Black Lightning units.

(a) Write the linear program we can use to plan our production to maximize total profit.
The resulting LP should look familiar. If it doesn't, re-read all the questions on this exam.

(b) The delivery driver offers to sell us a few extra logic circuits for $2.50 each. Should we buy some? Why? What if the asking price is $3.50 each?

(c) The retailer who buys our products telephones with a nasty surprise: paranoid hackers are scaring away respectable customers. The retailer insists that $x_2 \leq x_1$. Show how to use the Dual Simplex Method to determine our revised production plan when this constraint is added to our problem. What happens to our profit?

(d) Part (c) was just a bad dream. Forget it. This morning we have good news. Our co-op student has figured out how to make a pocket music player using 2 logic circuits, 1 crypto chip, and 2 memory chips. What value(s) of profit-per-unit on music players would motivate us to change our current plan and make some? How will our overall production strategy change in this case?

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