THE UNIVERSITY OF BRITISH COLUMBIA
SESSIONAL EXAMINATIONS – APRIL 2017
MATHEMATICS 323

Time: 2 hours 30 minutes

Instructions: Total number of points: 100.

The mark for the exam will be based on Question 1 and the **best 4** of the remaining questions. That is, you need to do only Question 1 and any 4 of the remaining ones (provided you do them correctly) for full marks. You can attempt as many questions as you want.

You can use the statements we proved in class, or the theorems proved in the textbook, without proof; but you need to provide complete statements of all the results you quote.

Write your name and student number at the top of each booklet you use, and please number the booklets (e.g. "booklet 2 of 5") if you use more than one.

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1. Mandatory question [36 points] (each sub-part is 3 points).

Determine whether the following statements are **True** or **False** (you have to include short proofs/counterexamples):

(1) (a) Any prime element of $\mathbb{Z}[\sqrt{-3}]$ is irreducible.
(b) Any irreducible element of $\mathbb{Z}[\sqrt{-3}]$ is prime.

(2) (a) The ideal (3) is maximal in $\mathbb{Z}[i]$.
(b) The polynomial $f(x) = x^5 + 3ix^4 - 36x^3 + (9+6i)x - 12$ is irreducible in $\mathbb{Z}[i][x]$.
(c) The polynomial $f(x) = x^5 + 3ix^4 - 36x^3 + (9+6i)x - 12$ is irreducible in $\mathbb{Q}(i)[x]$.

(3) (a) The rings $\mathbb{H}$ (the quaternions with real coefficients) and $M_2(\mathbb{R})$ (real 2×2-matrices) are isomorphic.
(b) The $\mathbb{R}$-modules $\mathbb{H}$ and $M_2(\mathbb{R})$ are isomorphic.

(4) Recall that for a module M over an integral domain R, $\mathrm{Tor}(M)$ is the set of torsion elements of M; an element $m \in M$ is called torsion if there exists $r \in R$, $r \neq 0$, such that $r \cdot m = 0$.
(a) Let $R = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, and let M be a finitely generated module over R. Then $M/\operatorname{Tor}(M)$ is free.
(b) For any finitely generated module over $R = \mathbb{Z}[\sqrt{-5}]$, $M/\operatorname{Tor}(M)$ is free.

(5) Let R be an integral domain.
(a) Direct sum of cyclic R-modules is always cyclic.
(b) Direct sum of torsion R-modules is always torsion.
(c) Direct sum of free R-modules is always free.

The following questions are 16 points each. You only need to do any 4 of them for full marks.

2. (a) Is $x^4 + x + 1$ irreducible in $\mathbb{F}_2[x]$?
(b) Describe the quotient $\mathbb{F}_2[x]/(x^4 + x + 1)$.
(c) Is $f(x) = x^4 - 20x^3 + 6x^2 + 3x - 5$ irreducible in $\mathbb{Q}[x]$?
Hint: Use Part (a)

3. Let $I = (x^4 + 2x^2 + 1, x^3 - x^2 + x - 1)$ in $\mathbb{F}_3[x]$.
(a) Is the ideal I principal? If yes, find its generator.
(b) Describe the quotient $\mathbb{F}_3[x]/I$. Is I maximal?
(c) Let $\alpha = \bar{x}$ be the image of $x \in \mathbb{F}_3[x]$ in the quotient $\mathbb{F}_3[x]/I$. Find $(1+\alpha)^{-1}$ (that is, find an expression for $(1+\alpha)^{-1}$ in terms of elements of $\mathbb{F}_3$ and α).

4. (a) Classify all abelian groups of order 96.
(b) Recall that for an R-module M, $\mathrm{End}_R(M)$ is the set of all module homomorphisms from M to itself. It is also an R-module, and a ring (with respect to composition).
Find $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$.

5. Let R be an integral domain.
(a) Prove that for any R-module M, the module $M/\operatorname{Tor}(M)$ is torsion-free.
(b) Suppose an R-module M has a submodule N such that N is free and M/N is torsion. Does it follow that M is a direct sum of a free module and a torsion module?

6. Let N be the submodule of $\mathbb{Z}^3$ generated by the vectors $\langle 2, 3, 2\rangle$, $\langle 2, 2, 0\rangle$, and $\langle -5, -6, -2\rangle$.
(a) Find the quotient $\mathbb{Z}^3/N$.
(b) Is N free? If yes, find the rank of N.

7. (a) Prove that for any $f \in \mathbb{C}[x]$, $a \in \mathbb{C}$, the polynomial $f(x)$ can be rewritten as $f(x) = b_n(x-a)^n + b_{n-1}(x-a)^{n-1} + \ldots b_1(x-a) + b_0$, and $b_0 = f(a)$.
(b) Recall *Partial Fraction decomposition* that you learned in integral calculus in order to compute integrals of rational functions: for $f, g \in \mathbb{C}[x]$, if $g(x) = (x-a_1)^{k_1} \ldots (x-a_n)^{k_n}$, then

$$\frac{f(x)}{g(x)} = h(x) + \sum_{i=1}^{k_1} \frac{c_{i1}}{(x-a_1)^i} + \sum_{i=1}^{k_2} \frac{c_{i2}}{(x-a_2)^i} + \cdots + \sum_{i=1}^{k_n} \frac{c_{in}}{(x-a_n)^i},$$

where $h \in \mathbb{C}[x]$ is a polynomial, and $c_{ij} \in \mathbb{C}$.
Prove that the partial fraction decomposition exists.
Hint: Use Chinese Remainder Theorem for rings and Part (a).