The University of British Columbia. Mathematics 322

Final Examination - Monday, December 10, 2012, 3:30-6pm. Instructor: Reichstein

Last Name _____ **First** _____ **Signature** _____

Student Number _____

Every problem is worth 5 points.

Rules governing examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

(a) speaking or communicating with other candidates, unless otherwise authorized;
(b) purposely exposing written papers to the view of other candidates or imaging devices;
(c) purposely viewing the written papers of other candidates;
(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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Total	

Problem 1: Let $f_1 \colon A \to B$ and $f_2 \colon B \to C$ be homomorphisms of groups.
(a) Show that the composition $f \colon A \to C$ defined by

$$f(a) = f_2(f_1(a))$$

for every $a \in A$, is also a homomorphism.
(b) If f is an isomorphism, can we conclude that both f_1 and f_2 are isomorphisms? If so, give a proof; if not, give a counterexample.

Problem 2: Let G be a group and H, K be subgroups of G. The product HK of H and K is defined as $\{hk \mid h \in H,\, k \in K\}$.

(a) Show that if H is normal in G then HK is a subgroup of G.

(b) Show that if $|H| \cdot |K| > |G|$ then $H \cap K \neq \{e\}$. Here H and K are not assumed to be normal in G.

Problem 3: Let G be a group, $Z(G)$ be the center of G and g be an element of G. Suppose that $C_G(g) = Z(G)$ for some element $g \in G$. (Here as usual, $C_G(g) = \{h \in G \,|\, gh = hg\}$ denotes the centralizer of g in G.) Show that then (a) $g \in Z(G)$ and (b) G is abelian.

Problem 4: Let $f\colon G \to H$ be a surjective homomorphism. Assume $|G| = 8|H|$. Show that
(a) if P is a Sylow 2-subgroup of G then $f(P)$ is a Sylow 2-subgroup of H.
(b) Conversely, show that every Sylow 2-subgroup Q of H is of the form $f(P)$ for some Sylow 2-subgroup P of G.

Problem 5: Let G be a finite group and N be a normal subgroup of G of index d. (Recall that this means that $d = \dfrac{|G|}{|N|}$.) If p is a prime and d is not divisible by p, show that every Sylow p-subgroup of G is contained in N.

Problem 6: How many Sylow 5-subgroups does the alternating group A_5 have?

Problem 7: How many abelian groups G of order 3^{50} are there (up to isomorphism), with the property that $a^9 = 1$ for every $a \in G$?

Problem 8: How many elements of order 7 does the group G have, (a) if $G = Z/(7^{99}\mathbb{Z})$? (b) if $G = (\mathbb{Z}/7\mathbb{Z})^{99}$? How many subgroups of order 7 does G have in each case?