**The University of British Columbia**

Final Examination - December 13, 2010

**Mathematics 322**

Closed book examination Time: 2.5 hours

**Last Name** ________________ **First** ____________ **Signature** ________________

**Student Number** __________________

## Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

PROBLEM 1. (3 pts) Let $G$ be a group and $S \subset G$ a subset. Define

$$N = \{g \in G | gSg^{-1} = S\}.$$

Prove that $N$ is a subgroup of $G$.

PROBLEM 2. (3 pts) Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial, such that $f(\alpha) = 0$ for some $\alpha \in \mathbb{Q}$. Prove that then $\alpha \in \mathbb{Z}$.

PROBLEM 3. (3 pts) Prove that $x^3 - 3x + 1$ is irreducible in $\mathbb{Q}[x]$.

PROBLEM 4. (3 pts) Let $U_n \subset \mathbb{Z}_n$ be the multiplicative group of units. Describe the groups $U_{10}$ and $U_8$. Are they isomorphic?

PROBLEM 5. (3 pts) Let $p, q \in \mathbb{Z}$ be distinct primes. Prove that for any $a \in \mathbb{Z}$ relatively prime to $pq$,

$$a^{(p-1)(q-1)} \equiv 1 \pmod{pq}.$$

PROBLEM 6. (5 pts) Let $D_{2n}$ be the dihedral group of order $2n$. Let $f : D_{12} \to D_6 \times \mathbb{Z}_2$ be defined by

$$f(\tau^i \sigma^j) = (\tau^i \sigma^j, [i + j]).$$

Here we represent elements of $D_{2n}$ as products $\tau^i \sigma^j$, where $i \in \{0, 1\}$, $j \in \{0, \ldots, n-1\}$. Multiplication is defined by $\tau^2 = e$, $\sigma^n = e$, and $\sigma\tau = \tau\sigma^{-1}$.

1. Prove that $f$ is a homomorphism of groups.
2. Prove that $f$ is an isomorphism.

PROBLEM 7. (3 pts) Is $x^5 + x + 1$ irreducible in $\mathbb{Z}_2[x]$? If it is reducible, factor it into irreducibles.

PROBLEM 8. (5 pts) Let $D_{2n}$ be the dihedral group of order $2n$.

1. Find the number of elements of order 2 in $D_{2n}$ for $n$ odd.
2. Prove that no two of the groups

$$D_{30}, \quad D_{10} \times \mathbb{Z}_3, \quad D_6 \times \mathbb{Z}_5$$

are isomorphic.

PROBLEM 9. (5 pts) Prove that there is no group homomorphism from $S_4$ onto $D_8$. (Hint: study the kernel of such a homomorphism.)

PROBLEM 10. (5 pts) Let $f, g \in \mathbb{Z}_5[x]$,

$$f(x) = x^2 - x - 2, \quad g(x) = x^3 - 2x + 1.$$

1.Find $\gcd(f, g)$.

2.Find a monic generator for the ideal $(f) \cap (g)$.

PROBLEM 11. (3 pts) Find the number of non-isomorphic abelian groups of order 108.

PROBLEM 12. (3 pts) Let $G$ and $H$ be finite groups of order $m$ and $n$, respectively, where $m$ and $n$ are relatively prime. If $f : G \to H$ is a group homomorphism, prove that $f(g) = e$ for every $g \in G$.

PROBLEM 13. (5 pts) Let $(Q, \cdot)$ be the group of quaternions.

$$Q = \{1, i, j, k, -1, -i, -j, -k\}.$$

The group operation is defined by:

$$i^2 = j^2 = k^2 = -1, \quad ij = k, jk = i, ki = j, \quad ji = -k, kj = -i, ik = -j, \quad -1 \cdot a = a \cdot (-1) = -a.$$

The element 1 is the identity element. You may assume that $Q$ is a group.

1.Find the center $Z(Q)$.

2.Describe the quotient group $Q/Z(Q)$. (Is it abelian? If so, which abelian group is it?)

PROBLEM 14. (5 pts) Let $I, J \subset R$ be two ideals in a (commutative) ring $R$. The product ideal $IJ$ is the set of all elements $r \in R$ that can be expressed as finite sums:

$$r = \sum_i a_i b_i, \qquad a_i \in I, b_i \in J.$$

You may assume that $IJ$ is an ideal in $R$.

Prove that if there exist $a \in I$ and $b \in J$ such that $a + b = 1$, then

$$IJ = I \cap J.$$

PROBLEM 15. (5 pts) Let $G$ be a group of order 255, such that the center $Z(G)$ is not the trivial one element group. Prove that $G$ is abelian. (Hint: this problem requires some knowledge of groups of order $pq$, where $p, q$ are primes.)

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