THE UNIVERSITY OF BRITISH COLUMBIA
SESSIONAL EXAMINATIONS – DECEMBER 2009
MATHEMATICS 322

Time: 2 hours 30 minutes

1. [16 points] Determine whether the following statements are true or false (you have to include proofs/counterexamples):
(a) The rings $\mathbb{Z}/35\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ are isomorphic.
(b) The groups $\mathbb{Z}/24\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ are isomorphic.
(c) If G is a cyclic group of order n, and $d|n$, then G has a subgroup of order d.
(d) The groups $\mathbb{F}_{p^2}^*$ and $(\mathbb{Z}/p^2\mathbb{Z})^*$ are isomorphic.

2. [13 points]
(a) Let G be a group, and N – a normal subgroup of G. Prove that if N contains an element g, then N contains the entire conjugacy class of g.
(b) Let σ be the following element of S_4:

$$\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{smallmatrix}\right).$$

Find the number of elements in the conjugacy class of σ.
(c) Prove that the permutation σ from Part (b) cannot be contained in any proper normal subgroup of S_4.

3. [8 points] Let M and N be normal subgroups of a group G. Suppose that $M \cap N = \{e\}$. Prove that for every $m \in M$ and $n \in N$, $mn = nm$.

4. [12 points] Find the set of units and the set of zero divisors in the ring R, where:
(a) R is the ring $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
(b) Let R is the quotient ring $\mathbb{Z}[\sqrt{7}]/I$, where I is the ideal

$$I = \{a + b\sqrt{7} \mid 6|a - b\}.$$

(Hint: find a convenient homomorphism from $\mathbb{Z}[\sqrt{7}]$ to $\mathbb{Z}/6\mathbb{Z}$).

5. [10 points] Let $R = \mathbb{F}_5[x]$; let $I = \langle x^2 + 1 \rangle$ be the ideal in R generated by the polynomial $x^2 + 1$, and let $J = \langle x^3 + 2 \rangle$ be the ideal generated by the polynomial $x^3 + 2$. Prove that $I + J$ is a principal ideal in R, and find its generator.

6. [10 points]
(a) Factor the element $5 \in \mathbb{Z}[i]$ as a product of irreducible elements.
(b) Is $\langle 5 \rangle$ a maximal ideal in $\mathbb{Z}[i]$?

7. [15 points]
(a) Prove that the polynomial $x^3 + 2x + 1$ is irreducible in $\mathbb{F}_3[x]$.
(b) Let I be the ideal $I = \langle x^3 + 2x + 1 \rangle$ in $\mathbb{F}_3[x]$, and let $R = \mathbb{F}_3[x]/I$. Let $\alpha = x + I \in R$. Prove that the element $1 + \alpha$ has an inverse in R.
(c) Find $(1 + \alpha)^{-1}$ (that is, find $\gamma \in R$, such that $\gamma(1 + \alpha) = 1$).

8. [8 points] Is $\mathbb{Z}[\sqrt{-3}]$ a principal ideal domain?

9. [8 points] Let G be a group acting on a set X. Suppose that the stabilizer G_x of a certain point $x \in X$ is a proper *normal* subgroup of G. Prove that every element of G_x fixes every point $y \in O_x$.

Extra credit problems:

1. Describe the quotient ring $\mathbb{Z}[i]/\langle 3\rangle$.

2. Let $G = \mathrm{GL}_n(\mathbb{F}_q)$ be the group of invertible $n \times n$-matrices with entries in $\mathbb{F}_q$.

(a) let $n = 2$, and consider the natural action of G on the set $\mathbb{F}_q^2 = \mathbb{F}_q \times \mathbb{F}_q$ defined by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

Find the stabilizer of the point $(1, 0) \in \mathbb{F}_q^2$.

(b) Find the order of $\mathrm{GL}_2(\mathbb{F}_q)$.

(c) Find the order of $\mathrm{GL}_n(\mathbb{F}_q)$.