Be sure that this examination has 12 pages including this cover

The University of British Columbia

Sessional Examinations - April 2017

Mathematics 321

Real Variables II

Closed book examination

Time: $2\frac{1}{2}$ hours

Name ____________________ **Signature** ____________________

Student Number____________________ **Instructor's Name** ___ Joel Feldman ___

Section Number _______ 201 _______

Special Instructions:

No calculators, notes, or other aids are allowed.

Rules Governing Formal Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

i. speaking or communicating with other examination candidates, unless otherwise authorized;

ii. purposely exposing written papers to the view of other examination candidates or imaging devices;

iii. purposely viewing the written papers of other examination candidates;

iv. using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

v. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) — (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		9
2		15
3		15
4		16
5		15
6		15
7		15
Total		100

Marks

[9] **1.** Define

(a) $\overline{\int_a^b} f(x)\, d\alpha(x)$

(b) equicontinuity

(c) convergence in the mean

Continued on page 3

[15] **2.** Answer True (with proof) or False (with specific counterexample):

(a) If f is monotonic on $[a, b]$, then f is Riemann integrable on $[a, b]$.

(b) Let f be Riemann integrable on $[0, 1]$. Then there is a $c \in [a, b]$ such that $f(c) = \int_0^1 f(t)\, dt$.

(c) Let f be Riemann integrable on $[a, b]$. Then the function $F : [a, b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f(t)\, dt$ is Riemann integrable on $[a, b]$.

(d) Let f be Riemann integrable on $[a, b]$ and let G be a differentiable function on $[a, b]$ with $G' = f$. Then $G(x) = \int_a^x f(t)\, dt$.

(e) Let f be a bounded real function on $[a, b]$ with f^2 Riemann integrable. Then f is Riemann integrable.

Continued on page 4

Continued on page 5

[15] **3.** Let $a < c < b$ and let $f : (a, b) \to \mathbb{R}$ be continuous on (a, b) and differentiable on (a, c) and on (c, b). Assume that $\lim_{x \to c} f'(x)$ exists. Prove that f is differentiable at $x = c$ and that $f'(x)$ is continuous at $x = c$.

Continued on page 6

[16] **4.** Let $g : [0, 1] \to \mathbb{R}$ be bounded, and $\alpha : [0, 1] \to \mathbb{R}$ be non–decreasing. Assume that, for every $\delta > 0$, the restriction of g to $[\delta, 1]$ is Riemann–Stieltjes integrable on $[\delta, 1]$ with respect to the restriction of α to $[\delta, 1]$.

(a) Prove that if α is continuous at 0, then g is Riemann–Stieltjes integrable with respect to α on $[0, 1]$.

(b) Give an example of functions g and α which shows that the conclusion of (a) may be false if the continuity of α at 0 is not assumed. You must prove that your function g is not Riemann–Stieltjes integrable with respect to α on $[0, 1]$.

Continued on page 7

Continued on page 8

[15] **5.** Let $\mathcal{M}$ be a metric space and let $\mathcal{F}$ be a bounded family of real valued functions on $\mathcal{M}$. Assume that $\mathcal{F}$ is equicontinuous. Define, for each $x \in \mathcal{M}$,

$$s(x) = \sup_{f \in \mathcal{F}} f(x)$$

Prove that s is continuous.

Continued on page 9

[15] **6.** Let $\alpha, \beta : [0,1] \to \mathbb{R}$ be non–decreasing continuous functions that obey $\alpha(0) = \beta(0)$. Assume that

$$\int_0^1 e^{-nx}\, d\alpha(x) = \int_0^1 e^{-nx}\, d\beta(x)$$

for all non–negative integers n.

(a) Prove that $\int_0^1 f(x)\, d\alpha(x) = \int_0^1 f(x)\, d\beta(x)$ for all continuous functions $f : [0,1] \to \mathbb{R}$.

(b) Does it follow that $\alpha(x) = \beta(x)$ for all $0 \le x \le 1$? You must justify your conclusion.

Continued on page 10

Continued on page 11

[15] **7.** Let f and f_n, $n \in \mathbb{N}$, be real–valued functions that are Riemann integrable on the interval $[a, b]$. Assume that

$$\int_a^b f_n(x)\, f_m(x)\; dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Prove that

$$\lim_{n\to\infty} \int_a^b f(x)\, f_n(x)\; dx = 0$$

Continued on page 12

The End