Math 321 Final Exam

8:30am, Tuesday, April 20, 2010

Duration: **150 minutes**

Name: ______________________ Student Number: ______________________

Do not open this test until instructed to do so! This exam should have 17 pages, including this cover sheet. **No** textbooks, calculators, or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam. **Circle your solutions! Reduce your answer as much as possible. Explain your work.** Use the back of the page if necessary.

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Notation/Definition: $\mathbb{R}$ = the set of real numbers; $\mathbb{C}$=the set of complex numbers; $\mathbb{N}$=the set of natural numbers; $\mathbb{Z}$=the set of integers; i=the imaginary number $\sqrt{-1}$; All Riemann integrable functions are bounded.

Problem	1	2	3	4	5	6	**Total**	Extra
Out of	16	12	10	8	10	14	70	5
Score								

Problem 1 (16 points). Let $\mathcal{E}$ be the space of 2π-periodic, continuously differentiable real-valued functions on $\mathbb{R}$ with $\int_{-\pi}^{\pi} f(x)dx = 0$; i.e.

$$\mathcal{E} = \{f : \mathbb{R} \to \mathbb{R} \mid f \in C^1 \text{ on } \mathbb{R}, \int_{-\pi}^{\pi} f(x)dx = 0, \text{ and } f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R}\}.$$

Let the norm $\|\cdot\|_{1,\infty}$ on $\mathcal{E}$ be defined as

$$\|f\|_{1,\infty} = \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}} |f'(x)|.$$

Let d_1 be a metric on $\mathcal{E}$ defined as $d_1(f,g) = \|f-g\|_{1,\infty}$, for $f, g \in \mathcal{E}$.

(a) (3 points) Recall that for $n \in \mathbb{Z}$ the Fourier coefficient $\widehat{f}(n)$ is given as $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$. Prove that for $f \in \mathcal{E}$, $|\widehat{f}(n)| \leq \|f\|_{1,\infty}$ and $|n||\widehat{f}(n)| \leq \|f\|_{1,\infty}$ for all $n \in \mathbb{Z}$.

(b) (6 points) Prove that the space $\mathcal{E}$ equipped with the metric d_1 is a complete metric space. (Hint: You may want to use the Fundamental Theorem of Calculus.)

(c) (7 points) Prove that for every $f \in \mathcal{E}$, its Fourier series converges **uniformly** to f on $\mathbb{R}$. In particular, $f(x) = \sum_{-\infty}^{\infty} \widehat{f}(n)e^{inx}$,

Problem 2 (**12 points**). Let $\mathcal{E}$ be the space of functions as given in **Problem 1**. You are allowed/encouraged to use the results/statements of **Problem 1**.

Let R be an operation on the functions in $\mathcal{E}$, defined as following: for $f \in \mathcal{E}$ written as the infinite sum $f(x) = \sum_{-\infty}^{\infty} \widehat{f}(n)e^{inx}$ *(this is possible for all $f \in \mathcal{E}$ because of* **Problem 1 (c)***)*, let the function $R[f]$ on $\mathbb{R}$ be defined as

$$R[f](x) = \sum_{-\infty}^{\infty} |\widehat{f}(100n)|^2 e^{inx} \qquad \text{for } x \in \mathbb{R}.$$

Of course, for this definition to work, we **need to prove** that the series expression converges. Verify this and furthermore prove that R gives a map from $\mathcal{E}$ to $\mathcal{E}$. Namely,

(a) (3 points) prove that the series $\sum_{-\infty}^{\infty} |\widehat{f}(100n)|^2 e^{inx}$ converges uniformly on $\mathbb{R}$;

(b) (2 points) prove that $R[f]$ is a **real**-valued function on $\mathbb{R}$. (Hint: $f \in \mathcal{E}$ is real-valued);

(c) (1 point) prove that $R[f]$ is 2π-periodic;

(d) (1 point) prove that $\int_{-\pi}^{\pi} R[f](x)dx = 0$;

(e) (5 points) prove that $R[f]$ is a continuously differentiable function on $\mathbb{R}$.

Problem 3 (10 points). Let $\mathcal{E}$ be the space of functions as given in **Problem 1** and let $K : \mathcal{E} \to \mathcal{E}$ be a mapping that satisfies for all $f, g \in \mathcal{E}$, $d_1(K[f], K[g]) \leq \frac{1}{2} d_1(f, g)$. Here, the metric d_1 is given in **Problem 1**. Assume that $K[0] = 0$. Here, 0 denotes the constant zero function. Let $F : \mathcal{E} \to \mathcal{E}$ be defined as $F[f] = f + K[f]$, for $f \in \mathcal{E}$.

(a) (3 points) Prove that F is injective, i.e. if $F[f] = F[g]$ then $f = g$.

(b) (7 points) Prove that for all $g \in \mathcal{E}$, with $\|g\|_{1,\infty} \leq \frac{1}{2}$,
there exists $f \in \mathcal{E}$, **with** $\|f\|_{1,\infty} \leq 1$, such that $F[f] = g$.

Problem 4 (8 points). Let $f : [0, 1] \to \mathbb{R}$ be a **Riemann integrable** function, such that $|f| \leq 1$ on $[0, 1]$. Suppose

$$\int_0^1 f(x)x^n dx = 0$$

for all $n \in \mathbb{Z}$, with $n \geq 0$. Let x_0 be a point in the interval $[0, 1]$. Assume that f is continuos at x_0.

Prove that $f(x_0) = 0$.

[Hint: From one of the HW problems, we know that this holds if f is continuous on $[0, 1]$. But, in this problem f is assumed to be continuous **only** at a fixed point x_0.)

Problem 5 (10 points). Let $n \in \mathbb{N}$. **Suppose** the function $F : \mathbb{R}^n \to \mathbb{R}^n$ has its **inverse** function $G : \mathbb{R}^n \to \mathbb{R}^n$, i.e. $F(G(x)) = x$ and $G(F(x)) = x$ for all $x \in \mathbb{R}^n$. Let $\mathbf{0}$ denote the origin in $\mathbb{R}^n$, i.e. $\mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{R}^n$. Assume $F(\mathbf{0}) = \mathbf{0}$ and F is differentiable at $x = \mathbf{0}$. Assume that the derivative $DF(\mathbf{0})$ of F at $x = \mathbf{0}$, is invertible. (Note that the derivative is given by an n-by-n matrix.)

Prove carefully using the definition of derivative of functions of several variables, that the function G is also differentiable at $\mathbf{0}$.

(Here is a minor hint: at certain moment, you may want to use the fact that for an n-by-n invertible matrix A, there exists a constant $\lambda > 0$ such that $|A\mathbf{u}| \geq \lambda$ for any vector $\mathbf{u} \in \mathbb{R}^n$ with $|\mathbf{u}| = 1$.)

Problem 6 (14 points). Let $i = \sqrt{-1}$ be the pure imaginary number. Let $\{f_n\}$ be the sequence of functions $f_n : \mathbb{R} \to \mathbb{C}$ defined as $f_n(x) = e^{inx}$ for $x \in \mathbb{R}$, i.e. $f_1(x) = e^{ix}, f_2(x) = e^{i2x}, f_3(x) = e^{i3x}, \cdots$.

Let $\{g_n\}$ be a sequence of 2π-**periodic** Riemann integrable functions $g_n : \mathbb{R} \to \mathbb{R}$. **Assume** that for all n, $|g_n(x) - g_n(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Consider the sequence of 2π-periodic functions $\{f_n * g_n\}$, where $f_n * g_n : \mathbb{R} \to \mathbb{C}$ is given by

$$f_n * g_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(x - t) g_n(t) dt \qquad \text{for } x \in \mathbb{R}.$$

(a) (**7 points**) Prove that there exists a **subsequence** $\{h_k\}$ **of** $\{f_n * g_n\}$ such that as $k \to \infty$, h_k converges uniformly to a function, say h_∞, on $\mathbb{R}$.

(Hint: Observe that for $n \geq 1$, $f_n * g_n = f_n * \big(g_n - g_n(0)\big)$.)

(b) (**5 points**) Let $\{u_k\}$ be a **sub**sequence of $\{f_n * g_n\}$ such that $\{u_k\}$ converges uniformly on $\mathbb{R}$. (*Such uniformly convergent subsequence exists by the result of* (a).) Prove that as $k \to \infty$, $u_k \to 0$ uniformly on $\mathbb{R}$, i.e.

$$\lim_{k \to \infty} \big[\sup_{x \in \mathbb{R}} |u_k(x)| \big] = 0.$$

(Hint: this problem could be hard.)

(c) (**2 points**) Use (a) (or the solution of (a)) and (b) to prove that in fact $f_n * g_n \to 0$ uniformly on $\mathbb{R}$.

Extra Problem (5 points). (This problem is only for extra mark, and it could be difficult. Do not try this unless you have time left **after** finishing all the previous problems.)

Let $\mathcal{E}$ be the space of functions given in **Problem 1**, and let $R : \mathcal{E} \to \mathcal{E}$ be the mapping given in **Problem 2**. Let $X = \{f \in \mathcal{E} \mid \|f\|_{1,\infty} \leq 1\}$, i.e X is the subset of $\mathcal{E}$ consisting of functions f with $\|f\|_{1,\infty} \leq 1$. **Prove** that R satisfies that for all $f, g \in X$,

$$d_1(R[f], R[g]) \leq \frac{1}{2} d_1(f, g).$$

Here, the metric d_1 is as given in **Problem 1**. You are allowed/encouraged to use the results/statements of **Problem 1** and **Problem 2** . (Hint: If you can solve **Problem 2 (e)**, it is likely that you can solve this problem. You may want to use the identity $|a|^2 - |b|^2 = (|a| + |b|)(|a| - |b|)$ and the inequality

$$\big||a| - |b|\big| \leq |a - b|$$

for complex numbers $a, b \in \mathbb{C}$.)