Be sure this exam has 14 pages including the cover

The University of British Columbia

Sessional Exams – April 2017
MATH 318 Probability with Physical Applications
Dr. G. Slade

Last Name: ______________________ **First Name:** ______________________

Student Number: ____

This exam consists of **8** questions worth **10** marks each.
No aids are permitted. There are tables on the last page.
Please show all work and calculations. Numerical answers need not be simplified.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

(2 marks) 1. (a) Two independent events E, F have probabilities $P(E) = 0.1$ and $P(F) = 0.3$. What are the probabilities that:

(i) at least one of the two events occurs?

(ii) exactly one of the two events occurs?

(3 marks) (b) Two cards are dealt from a deck of 52. What is the probability that the second card is a spade, given that the first card is black?

(2 marks) (c) A total of ten black balls and ten white balls are randomly divided between urn #1 and urn #2 in such a way that each urn contains ten balls. What is the probability that urn #1 contains exactly three white balls?

(3 marks) (d) Three boys and three girls are seated in a random order in a row. What is the probability that boys and girls alternate in the row?

2. A manufacturing process produces integrated circuit chips. The long run fraction of bad chips produced is 20%. It is expensive to thoroughly test a chip to determine whether it is good or bad, and a cheap but less effective test is used instead. All good chips will pass the cheap test, but so will 10% of the bad chips.

(8 marks) (a) What proportion of chips that pass the cheap test are in fact good chips?

(2 marks) (b) If all chips that pass the cheap test are sold, what percentage of chips sold will be bad chips?

(10 marks) 3. An astrophysicist performs an experiment to measure the distance d in light years between two stars, but her measurements are not completely accurate due to experimental errors. Suppose that each measurement is an independent random variable with mean equal to the true distance d and variance 9 (light years)^2. She uses the average of her measurements as an estimate for d. Using the central limit theorem, determine the number of measurements needed to be 95% confident that the estimated distance is within ± 1 light year of the true distance.

4. Let X, Y be independent exponential random variables with parameter λ.

(2 marks) (a) Compute the probability density function of $X + Y$.

(2 marks) (b) Determine the characteristic function of $X + Y$.

(2 marks) (c) Compute $P(X < 4Y)$.

(2 marks) (d) Determine the expected value of X given that $Y > 1$, i.e., determine $E[X|Y > 1]$.

(2 marks) (e) Determine the expected value of X given that $X > 1$, i.e., determine $E[X|X > 1]$.

5. Customers arrive at a server according to a Poisson process of rate 3 per minute.

(2 marks) (a) Find the probability that no customer arrives during the first 30 seconds.

(3 marks) (b) Find the probability that exactly three customers arrived during the first two minutes, exactly one of whom arrived during the first minute.

(2 marks) (c) Find the probability that the fourth customer arrives within 30 seconds of the third customer.

(3 marks) (d) Find the probability that the fifth customer takes more than two minutes to arrive.

6. The random walk on the 3-dimensional body-centred cubic lattice is the random walk on $\mathbb{Z}^3$ that takes steps $(1,1,1)$, $(1,1,-1)$, $(1,-1,1)$, $(1,-1,-1)$, $(-1,1,1)$, $(-1,1,-1)$, $(-1,-1,1)$, $(-1,-1,-1)$ with equal probabilities $\frac{1}{8}$.

(3 marks) (a) Determine the characteristic function $\phi_1(k_1, k_2, k_3)$ of a single step. Simplify your answer as much as possible.

(2 marks) (b) Identify *all* points (k_1, k_2, k_3) where $\phi_1(k_1, k_2, k_3) = 0$.

(3 marks) (c) The integrability of $\frac{1}{1-\phi_i(\vec{k})}$ turns out to the be the same at each of the points in part (b) (you need not verify this). Determine (give details) whether the integral of $\frac{1}{1-\phi_i(\vec{k})}$ near $\vec{k} = (0, 0, 0)$ is finite or infinite.

(2 marks) (d) Is the random walk on the body-centred cubic lattice recurrent or transient? Explain.

7. An urn contains 8 balls, some of them white and some of them black. At each time step, a ball is chosen from the urn at random. With probability p the chosen ball is replaced by a white ball, and with probability $1-p$ it is replaced by a black ball. The state of the system is the number of white balls in the urn. This defines a Markov chain.

(2 marks) (a) Determine all communicating classes, their periods, and their transience or recurrence.

(3 marks) (b) Determine the transition probabilities P_{ij}.

(5 marks) (c) Using any method (and justifying your answer), find the proportion of time spent by the Markov chain in each state.

8. Smith has three books B_1, B_2, B_3 which he keeps on a shelf when he is not reading. After he reads a book, he returns it to the left of the other books on the shelf. His favourite book is B_1, which he selects to read with probability $\frac{2}{3}$. He selects B_2 with probability $\frac{1}{6}$, and he selects book B_3 also with probability $\frac{1}{6}$. This defines a Markov chain, with the state of the system given by the different possible orders for the books on the shelf. In your solution, order your states in the order: 123, 132, 213, 231, 312, 321.

(3 marks) (a) Write down the transition matrix for the Markov chain.

(6 marks) (b) Determine the stationary distribution of the Markov chain.

(1 marks) (c) Suppose the books are now in order $B_3B_2B_1$. How many times will Smith select a book again, on average, until the books are again in the same order?

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial (n, p)	np	$np(1-p)$	$(1-p+pe^{it})^n$
Geometric (p)	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson (λ)	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{ita}-e^{itb}}{it(b-a)}$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda-it}$
Normal (μ, σ^2)	μ	σ^2	$e^{i\mu t-\sigma^2 t^2/2}$

Table 2: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990