

Be sure this exam has 11 pages including the cover

The University of British Columbia

Sessional Exams – 2007/2008 Winter Term 2
Mathematics 318 Probability with Physical Applications, All sections
D. Brydges, M. Merle

Name: _____

Student Number: _____

This exam consists of 8 questions worth 10 marks each. No aids other than calculators are permitted.

Problem	total possible	score
1.	12	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	8	
total	80	

- 1. Each candidate should be prepared to produce his library/AMS card upon request.**
- 2. Read and observe the following rules:**
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
- (a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
 - (b) Speaking or communicating with other candidates.
 - (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- 3. Smoking is not permitted during examinations.**

Tables on last pages.

- (2 points) 1. (a) How many different seven letter License Plates can be made out of the letters in “hrududu”?
- (3 points) (b) What is the probability that a poker hand will have two of a kind and three of a kind? (e.g. two Kings and three Queens).
- (4 points) (c) Tommy has three cards. Two of them are white on one side and black on the other. The third is white on both sides. Tommy picks one of them at random and places it on a table with a white side upwards. What is the probability that the other side is white?
- (3 points) (d) A drunken postman has three letters to deliver to three of the four families living at an apartment block. He randomly puts one letter in each of three of the four mailboxes. What is the probability at least one letter goes to the correct destination? Hint. Inclusion/exclusion.

- (10 points) 2. Fifty numbers are rounded off to the nearest integer and then summed. If the individual round-off errors are uniformly distributed over $(-.5, .5)$, approximate the probability that the resulting sum differs from the exact sum by more than 3.

- (10 points) 3. A Boeing 747 carries 416 passengers. Airlines find that each passenger who reserves a seat fails to show up with probability 0.01 independently of other passengers. Anticipating no-shows the company operating the Boeing 747 sells 420 reservations for every flight. On approximately what proportion of flights will there be more passengers than seats?

4. Suppose that the #4 bus arrivals at Memorial Gym are distributed according to a Poisson process with rate $\lambda_4 = \frac{1}{20}$ (buses per minute). Likewise the #9 and #17 bus arrivals are Poisson processes with rates $\lambda_9 = \lambda_{17} = \frac{1}{10}$. Furthermore the three processes are independent.

(4 points)

- (a) What is the probability that none of these buses arrive in an interval of 5 minutes?

(3 points)

- (b) Little Tommy and his Dad arrive at Memorial Gym. Because Tommy likes seeing buses he makes his father wait until he has seen at least one #4, at least one #9 and at least one #17. What is the probability Dad has to wait 20 minutes?

(3 points)

- (c) Find the probability that a #4 arrives before either a #9 or a #17?

- (2 points) 5. (a) Suppose that a random walk is recurrent and that there is a positive probability for the walk to eventually visit some other site x , starting from the origin. Is x recurrent? Explain.

- (3 points) (b) Consider a random walk on \mathbb{Z}^1 taking steps 0, +2, -2 with respective probabilities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$. What is $\phi(k)$ in the formula

$$\text{expected \#visits to origin} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - \phi(k)} dk$$

- (2 points) (c) Is the walk defined in the previous part recurrent? Explain.

- (3 points) (d) Consider two independent walkers performing simple random walk (each taking steps ± 1 with equal probabilities $\frac{1}{2}$) on \mathbb{Z}^1 , with one walk beginning at 0 and the other at +2. Will the two walkers certainly meet? How is this related to the previous parts of this question?

6. Smith is playing gamblers ruin with the US Federal Reserve which has an infinite amount of money. Recall that in gamblers ruin Smith repeatedly plays a game in which with probability p he wins one dollar from the bank and with probability $q = 1 - p$ he loses one dollar to the bank. Smith plays until he is broke.

(2 points)

- (a) Write the recursion relation and the boundary condition(s) for the probability p_i that Smith goes broke starting with $\$i$.

(2 points)

- (b) Find the general solution to the recursion and boundary conditions of the previous part when $p = \frac{1}{2}$.

(2 points)

- (c) Suppose Smith starts with $\$10^{10}$. What is the probability he goes broke? Explain your answer.

(2 points)

- (d) Write the recursion relation and boundary condition(s) for the expected number of games M_i that Smith plays before going broke.

(2 points)

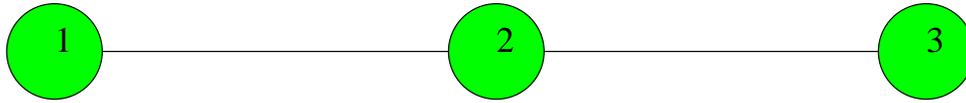
- (e) For $p = \frac{1}{2}$, find all solutions to the recursion relation and boundary condition(s) of the previous part. Deduce that M_i must be infinite by identifying a contradiction.

7. Consider the Markov chain $X_{n=0,1,\dots}$ with state space $\{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & \frac{1}{2} & \frac{1}{2} \\ 3 & 1 & 0 & 0 \end{array}$$

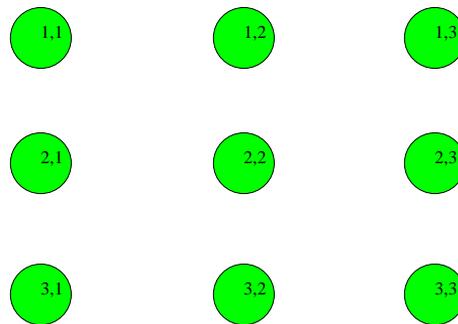
- (3 points) (a) What property of $(\mathbf{P}^n)_{11}$ must be checked to show that state 1 is aperiodic? Is state 1 aperiodic? Explain, using a transition diagram.
- (4 points) (b) Suppose $X_0 = 1$. In the long run what proportion of time does the Markov chain spend in state 2.
- (3 points) (c) Consider the transitions X_0 to X_1 , X_2 to X_3 etc. In the long run what proportion of these transitions will be from state 2 to state 3?

8. Consider a random walk $X_{n=0,1,\dots}$ on the graph shown. When the walk is at state 2 it chooses state 1 with probability $\frac{1}{2}$ and state 3 with probability $\frac{1}{2}$. When it is in state 1 or state 3 it always goes to state 2.



- (4 points) (a) What is the expected time to return, if the walk starts in state 1?

- (2 points) (b) Let $Y_{n=0,1,\dots}$ be an independent walk with the same transition probabilities as $X_{n=0,1,\dots}$ and consider (X_n, Y_n) as a new Markov chain on the 9 vertices pictured below. Complete the picture to a graph which shows the *positive probability* transitions.



- (2 points) (c) If both walks start in state 1, what is the expected time until they are again simultaneously in state 1?

Table 1: Mean and Variances

Distribution	Mean	Variance
Bin (n, p)	np	$np(1 - p)$
Geometric (p)	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson (λ)	λ	λ
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exp (λ)	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Table 2: cdf of normal distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

PERCENTAGE POINTS, CHI-SQUARE DISTRIBUTION (Continued)

$$P(x^2) = \int_0^{x^2} \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} x^{\frac{n-2}{2}} e^{-\frac{x^2}{2}} dx$$

$n \backslash P$.005	.010	.025	.050	.100	.250	.500	.750	.900	.950	.975	.990	.995
1	.0000393	.000157	.000982	.00393	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

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Chi Square Distribution

$n \backslash P$.80	.75	.70	.65	.60	.55	.50	.45	.40	.35	.30	.25	.20	.15	.10	.05	.025	.01	.005
1	.425	1.000	3.078	6.314	12.706	31.821	63.687	106.919	153.303	197.927	243.015	289.624	336.756	384.291	432.000	480.000	528.000	576.000	624.000
2	.280	.816	1.886	2.920	4.303	6.065	8.025	10.000	12.000	14.000	16.000	18.000	20.000	22.000	24.000	26.000	28.000	30.000	32.000
3	.227	.765	1.638	2.353	3.182	4.541	6.251	8.343	10.851	13.924	17.763	22.485	28.191	34.164	40.456	47.153	54.287	61.902	70.000
4	.201	.741	1.533	2.132	2.776	3.747	5.000	6.626	8.718	11.459	15.086	19.813	25.768	32.909	40.289	48.000	56.153	64.779	73.900
5	.187	.727	1.476	2.015	2.571	3.365	4.432	5.808	7.591	9.778	12.901	17.024	22.461	29.143	36.756	45.000	53.983	63.500	73.500
6	.185	.718	1.440	1.943	2.447	3.143	4.107	5.369	6.998	9.100	11.900	15.900	21.100	27.500	35.000	43.000	51.500	60.500	70.000
7	.183	.711	1.415	1.885	2.365	2.998	3.899	5.008	6.500	8.400	10.900	14.400	19.000	24.500	31.000	38.000	45.500	53.500	62.000
8	.182	.706	1.397	1.860	2.306	2.896	3.700	4.800	6.200	8.000	10.400	13.800	18.400	23.900	30.400	37.400	44.900	52.900	61.400
9	.181	.703	1.383	1.833	2.262	2.821	3.521	4.500	5.800	7.500	9.800	12.800	17.400	22.900	29.400	36.400	43.900	51.900	60.400
10	.180	.700	1.372	1.812	2.228	2.754	3.400	4.400	5.600	7.300	9.600	12.600	17.200	22.700	29.200	36.200	43.700	51.700	60.200
11	.180	.697	1.363	1.796	2.201	2.718	3.300	4.300	5.500	7.200	9.500	12.500	17.100	22.600	29.100	36.100	43.600	51.600	60.100
12	.180	.695	1.356	1.782	2.179	2.681	3.250	4.250	5.400	7.100	9.400	12.400	17.000	22.500	29.000	36.000	43.500	51.500	60.000
13	.180	.694	1.350	1.771	2.160	2.650	3.210	4.210	5.350	7.050	9.350	12.350	16.950	22.450	28.950	35.950	43.450	51.450	59.950
14	.180	.692	1.345	1.761	2.145	2.624	3.177	4.177	5.300	6.950	9.250	12.250	16.900	22.400	28.900	35.900	43.400	51.400	59.900
15	.180	.691	1.341	1.753	2.131	2.602	3.147	4.147	5.250	6.900	9.200	12.200	16.850	22.350	28.850	35.850	43.350	51.350	59.850
16	.180	.689	1.337	1.746	2.120	2.583	3.121	4.121	5.200	6.850	9.150	12.150	16.800	22.300	28.800	35.800	43.300	51.300	59.800
17	.180	.689	1.333	1.740	2.110	2.567	3.098	4.098	5.150	6.800	9.100	12.100	16.750	22.250	28.750	35.750	43.250	51.250	59.750
18	.180	.688	1.330	1.734	2.101	2.552	3.078	4.078	5.100	6.750	9.050	12.050	16.700	22.200	28.700	35.700	43.200	51.200	59.700
19	.180	.687	1.328	1.729	2.093	2.539	3.061	4.061	5.050	6.700	9.000	12.000	16.650	22.150	28.650	35.650	43.150	51.150	59.650
20	.180	.687	1.325	1.725	2.086	2.528	3.045	4.045	5.000	6.650	8.950	11.950	16.600	22.100	28.600	35.600	43.100	51.100	59.600
21	.180	.686	1.323	1.721	2.080	2.518	3.031	4.031	4.950	6.600	8.900	11.900	16.550	22.050	28.550	35.550	43.050	51.050	59.550
22	.180	.686	1.321	1.717	2.074	2.508	3.019	4.019	4.900	6.550	8.850	11.850	16.500	22.000	28.500	35.500	43.000	51.000	59.500
23	.180	.685	1.319	1.714	2.069	2.500	3.007	4.007	4.850	6.500	8.800	11.800	16.450	21.950	28.450	35.450	42.950	50.950	59.450
24	.180	.685	1.318	1.711	2.064	2.492	3.002	4.002	4.800	6.450	8.750	11.750	16.400	21.900	28.400	35.400	42.900	50.900	59.400
25	.180	.684	1.316	1.708	2.060	2.485	2.992	4.000	4.750	6.400	8.700	11.700	16.350	21.850	28.350	35.350	42.850	50.850	59.350
26	.180	.684	1.315	1.706	2.056	2.479	2.981	4.000	4.700	6.350	8.650	11.650	16.300	21.800	28.300	35.300	42.800	50.800	59.300
27	.180	.684	1.314	1.703	2.052	2.473	2.971	4.000	4.650	6.300	8.600	11.600	16.250	21.750	28.250	35.250	42.750	50.750	59.250
28	.180	.683	1.313	1.701	2.048	2.467	2.962	4.000	4.600	6.250	8.550	11.550	16.200	21.700	28.200	35.200	42.700	50.700	59.200
29	.180	.683	1.311	1.699	2.045	2.462	2.953	4.000	4.550	6.200	8.500	11.500	16.150	21.650	28.150	35.150	42.650	50.650	59.150
30	.180	.683	1.310	1.697	2.042	2.457	2.944	4.000	4.500	6.150	8.450	11.450	16.100	21.600	28.100	35.100	42.600	50.600	59.100
40	.180	.681	1.303	1.684	2.021	2.423	2.904	4.000	4.400	6.100	8.400	11.400	16.000	21.500	28.000	35.000	42.500	50.500	59.000
50	.180	.679	1.296	1.671	2.000	2.390	2.860	4.000	4.300	6.050	8.350	11.350	15.900	21.400	27.900	34.900	42.400	50.400	58.900
60	.180	.679	1.289	1.658	1.980	2.358	2.817	4.000	4.200	6.000	8.300	11.300	15.800	21.300	27.800	34.800	42.300	50.300	58.800
120	.180	.674	1.282	1.645	1.960	2.326	2.756	4.000	4.100	5.950	8.250	11.250	15.700	21.200	27.700	34.700	42.200	50.200	58.700

This table gives values of t such that

$$P(t) = \int_0^t \frac{1}{\sqrt{\pi n} \Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} dx$$

for n , the number of degrees of freedom, equal to 1, 2, ..., 30, 40, 60, 120, ∞ and for $P(t) = 0.00, 0.05, 0.10, 0.25, 0.50, 0.75, 0.90, 0.95, 0.975, 0.990, 0.995$. The t -distribution is symmetrical, so that $P(-t) = 1 - P(t)$.