Be sure this exam has 11 pages including the cover

The University of British Columbia

Sessional Exams – 2007/2008 Winter Term 2
Mathematics 318 Probability with Physical Applications, All sections
D. Brydges, M. Merle

Name: ______________________________

Student Number: ______________________________

This exam consists of **8** questions worth **10** marks each. No aids other than calculators are permitted.

Problem	total possible	score
1.	12	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	8	
total	80	

Tables on last pages.

(2 points) 1. (a) How many different seven letter License Plates can be made out of the letters in "hrududu"?

(3 points) (b) What is the probability that a poker hand will have two of a kind and three of a kind? (e.g. two Kings and three Queens).

(4 points) (c) Tommy has three cards. Two of them are white on one side and black on the other. The third is white on both sides. Tommy picks one of them at random and places it on a table with a white side upwards. What is the probability that the other side is white?

(3 points) (d) A drunken postman has three letters to deliver to three of the four families living at an apartment block. He randomly puts one letter in each of three of the four mailboxes. What is the probability at least one letter goes to the correct destination? Hint. Inclusion/exclusion.

(10 points) 2. Fifty numbers are rounded off to the nearest integer and then summed. If the individual round-off errors are uniformly distributed over $(-.5, .5)$, approximate the probability that the resulting sum differs from the exact sum by more than 3.

(10 points) 3. A Boeing 747 carries 416 passengers. Airlines find that each passenger who reserves a seat fails to show up with probability 0.01 independently of other passengers. Anticipating no-shows the company operating the Boeing 747 sells 420 reservations for every flight. On approximately what proportion of flights will there be more passengers than seats?

4. Suppose that the #4 bus arrivals at Memorial Gym are distributed according to a Poisson process with rate $\lambda_4 = \frac{1}{20}$ (buses per minute). Likewise the #9 and #17 bus arrivals are Poisson processes with rates $\lambda_9 = \lambda_{17} = \frac{1}{10}$. Furthermore the three processes are independent.

(4 points) (a) What is the probability that none of these buses arrive in an interval of 5 minutes?

(3 points) (b) Little Tommy and his Dad arrive at Memorial Gym. Because Tommy likes seeing buses he makes his father wait until he has seen at least one #4, at least one #9 and at least one #17. What is the probability Dad has to wait 20 minutes?

(3 points) (c) Find the probability that a #4 arrives before either a #9 or a #17?

(2 points) 5. (a) Suppose that a random walk is recurrent and that there is a positive probability for the walk to eventually visit some other site x, starting from the origin. Is x recurrent? Explain.

(3 points) (b) Consider a random walk on $\mathbb{Z}^1$ taking steps 0, +2, −2 with respective probabilities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$. What is $\phi(k)$ in the formula

$$\text{expected \#visits to origin} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-\phi(k)} dk$$

(2 points) (c) Is the walk defined in the previous part recurrent? Explain.

(3 points) (d) Consider two independent walkers performing simple random walk (each taking steps ± 1 with equal probabilities $\frac{1}{2}$) on $\mathbb{Z}^1$, with one walk beginning at 0 and the other at +2. Will the two walkers certainly meet? How is this related to the previous parts of this question?

6. Smith is playing gamblers ruin with the US Federal Reserve which has an infinite amount of money. Recall that in gamblers ruin Smith repeatedly plays a game in which with probability p he wins one dollar from the bank and with probability $q = 1 - p$ he loses one dollar to the bank. Smith plays until he is broke.

(2 points) (a) Write the recursion relation and the boundary condition(s) for the probability p_i that Smith goes broke starting with $\$i$.

(2 points) (b) Find the general solution to the recursion and boundary conditions of the previous part when $p = \frac{1}{2}$.

(2 points) (c) Suppose Smith starts with $\$10^{10}$. What is the probability he goes broke? Explain your answer.

(2 points) (d) Write the recursion relation and boundary condition(s) for the expected number of games M_i that Smith plays before going broke.

(2 points) (e) For $p = \frac{1}{2}$, find all solutions to the recursion relation and boundary condition(s) of the previous part. Deduce that M_i must be infinite by identifying a contradiction.

7. Consider the Markov chain $X_{n=0,1,..}$ with state space $\{1, 2, 3\}$ and transition matrix

$\mathbf{P}$ =

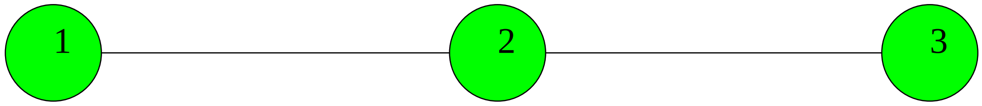
	1	2	3
1	0	1	0
2	0	$\frac{1}{2}$	$\frac{1}{2}$
3	1	0	0

(3 points) (a) What property of $(\mathbf{P}^n)_{11}$ must be checked to show that state 1 is aperiodic? Is state 1 aperiodic? Explain, using a transition diagram.

(4 points) (b) Suppose $X_0 = 1$. In the long run what proportion of time does the Markov chain spend in state 2.

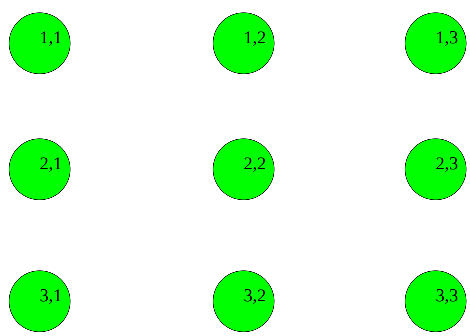
(3 points) (c) Consider the transitions X_0 to X_1, X_2 to X_3 etc. In the long run what proportion of these transitions will be from state 2 to state 3?

8. Consider a random walk $X_{n=0,1,\ldots}$ on the graph shown. When the walk is at state 2 it chooses state 1 with probability $\frac{1}{2}$ and state 3 with probability $\frac{1}{2}$. When it is in state 1 or state 3 it always goes to state 2.



(4 points) (a) What is the expected time to return, if the walk starts in state 1?

(2 points) (b) Let $Y_{n=0,1,\ldots}$ be an independent walk with the same transition probabilities as $X_{n=0,1,\ldots}$ and consider (X_n, Y_n) as a new Markov chain on the 9 vertices pictured below. Complete the picture to a graph which shows the *positive probability* transitions.



(2 points) (c) If both walks start in state 1, what is the expected time until they are again simultaneously in state 1?

Table 1: Mean and Variances

Distribution	Mean	Variance
Bin (n, p)	np	$np(1-p)$
Geometric (p)	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson (λ)	λ	λ
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exp (λ)	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Table 2: cdf of normal distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

PERCENTAGE POINTS, CHI-SQUARE DISTRIBUTION (Continued)

$$F(\chi^2) = \int_0^{\chi^2} \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n-2}{2}} e^{-\frac{x}{2}}\, dx$$

n \ F	.005	.010	.025	.050	.100	.250	.500	.750	.900	.950	.975	.990	.995
1	.0000393	.000157	.000982	.00393	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	.4.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

PERCENTAGE POINTS, STUDENT'S t-DISTRIBUTION

This table gives values of t such that

$$F(t) = \int_{-\infty}^{t} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} dx$$

for n, the number of degrees of freedom, equal to 1, 2, . . . , 30, 40, 60, 120, ∞; and for $F(t) = 0.60, 0.75, 0.90, 0.95, 0.975, 0.99, 0.995$, and 0.9995. The t-distribution is symmetrical, so that $F(-t) = 1 - F(t)$.

n \ F	.60	.75	.90	.95	.975	.99	.995	.9995
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	.289	.816	1.886	2.920	4.303	6.965	9.925	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	5.408
8	.262	.706	2.397	1.860	2.306	2.896	3.355	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	3.291