Be sure this exam has 10 pages including the cover

The University of British Columbia

Sessional Exams – 2006 Term 2
Mathematics 318 Probability with Physical Applications, All sections
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Last Name: ____________________ **First Name:** ____________________

Student Number: ____

Circle section number: 201 (Slade), 202 (Holroyd)

This exam consists of **8** questions worth **10** marks each. No aids are permitted.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

Tables on last page.

1. A closet contains 10 different pairs of shoes (each pair consists of a left shoe and a right shoe, so there are 20 shoes in total). 6 shoes are chosen at random. Find (do NOT simplify) the probability that:

(3 points) (a) 3 complete pairs are chosen;

(3 points) (b) exactly 1 complete pair is chosen;

(4 points) (c) at least one left shoe and at least one right shoe are chosen.

2. A certain coin lands on its edge with probability 1/1000 each time it is tossed. Carefully explain your answers to the following. Give ACTUAL NUMERICAL ANSWERS, using (if necessary) the approximations $\pi \approx 3.1$; $\ln 2 \approx 0.69$; $e^{-1} \approx 0.37$.

(2 points) (a) What is the expected number of tosses until the first time the coin lands on its edge?

(2 points) (b) How many times must the coin be tossed so that the expected number of times it lands on its edge is 1?

(3 points) (c) How many times must the coin be tossed so that the probability it lands on its edge at least once is approximately 1/2?

(3 points) (d) How many times must the coin be tossed so that the probability it lands on its edge at least 1000 times is approximately 1/2?

3. Let $X_1, X_2, \ldots$ be i.i.d. continuous random variables with probability density function

$$f(y) = \begin{cases} cy^{-4} & \text{if } |y| \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant. Let $S_0 = 0$ and let $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Then S_n denotes the position, after n steps, of a random walk performed on the real line (not just the integers), with steps X_i.

(2 points) (a) Determine the value of the constant c.

(2 points) (b) Determine the variance of a single step X_i.

(6 points) (c) Calculate the approximate probability that the walker is at least distance 60 from the origin after 300 steps, i.e., find $P(|S_{300}| \geq 60)$.

4. Pizza orders for meat and vegetarian pizzas arrive according to independent Poisson processes with rates μ and ν, respectively. Let M_s denote the number of meat orders, and V_s the number of vegetarian orders, received in the time interval $[0, s]$. Let $X_s = M_s + V_s$ denote the total number of orders received during $[0, s]$.

(2 points) (a) Write down the characteristic functions of M_s and V_s.

(2 points) (b) Hence find the characteristic function of $X_s = M_s + V_s$.

(1 points) (c) What is the distribution of X_s (give the name and parameter(s))?

(4 points) (d) Suppose that $M_1 = 1$, and let T denote the arrival time of the unique meat order during $[0, 1]$. Calculate the conditional probability $P(T \leq s|M_1 = 1)$, for $s \in [0, 1]$.

(1 points) (e) What is the conditional probability density function of T in part (a), given that $M_1 = 1$.

5. A certain coin comes up Heads with an unknown probability p each time it is tossed. It is tossed 6 times, giving 5 Heads.

(3 points) (a) Find an unbiased estimate for p, giving your reasoning.

(3 points) (b) Assuming the coin is fair (i.e. $p = 1/2$), what is the probability that the number of Heads is not in the range $\{2, 3, 4\}$?

(2 points) (c) Do the data provide grounds to reject the hypothesis that the coin is fair at the 10% level?

(2 points) (d) Using the same coin, suppose that 10 other people each toss the coin 6 times, and perform the same hypothesis test as above. If the coin is in fact fair, what is the probability at least one person will conclude that it is not fair? (Do not simplify your answer.)

6. Consider the Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c|cccccc|} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & \frac{1}{5} & \frac{3}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

(2 points) (a) Draw the transition diagram showing the six states with arrows indicating possible transitions and their probabilities.

(3 points) (b) Which states are aperiodic and which are periodic?

(3 points) (c) Which states are recurrent and which are transient?

(2 points) (d) Determine the probability, starting from state 0, of ever hitting state 5.

7. Two white balls and two black balls are divided between two urns, with each urn containing two balls. At each step, a ball is chosen at random from each urn, and the two balls are interchanged. (I.e., the ball from the first urn is placed in the second urn, and the ball from the second urn is placed in the first.) Let X_n denote the number of white balls in urn 1 after the nth step.

(1 points) (a) Explain briefly why (X_n) is a Markov chain.

(2 points) (b) Find the transition matrix $\mathbf{P}$.

(2 points) (c) Find the matrix of two-step transition probabilities.

(3 points) (d) Find the stationary distribution.

(1 points) (e) In the long run, what fraction of the time does urn 1 contain no white balls?

(1 points) (f) If there were m black and m white balls, with m balls per urn, and the same procedure was performed, guess the stationary distribution. (You do not need to verify that your guess is correct.)

8. A taxi is located either at the airport or in the city. From the city, the next trip is to the airport with probability 1/4, or to somewhere else in the city with probability 3/4. From the airport, the next trip is always to the city.

(2 points) (a) Set up a Markov Chain to model this situation, and write down its transition matrix.

(3 points) (b) Find the stationary distribution.

(2 points) (c) If the taxi starts at the airport, what is the expected number of trips until its next visit to the airport?

(3 points) (d) Suppose that each trip to or from the airport produces a profit of \$10, while each trip within the city produces an average profit of \$5. In the long run, what is the taxi's average profit per trip?

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial (n,p)	np	$np(1-p)$	$(1-p+pe^{it})^n$
Geometric (p)	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson (λ)	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform (a,b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{ita}-e^{itb}}{it(b-a)}$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda-it}$
Normal (μ,σ^2)	μ	σ^2	$e^{i\mu t-\sigma^2t^2/2}$

Table 2: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990