Be sure this exam has 10 pages including the cover

The University of British Columbia

Sessional Exams – 2005 Term 2
Mathematics 318 Probability with Physical Applications, All sections
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Name: ______________________________

Student Number: ______________________________

This exam consists of **8** questions worth **10** marks each. No aids other then calculators are permitted.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

Tables on last page.

(2 points) 1. (a) How many different 7-place license plates are possible if the first two places are for letters and the other 5 are for digits?

(3 points) (b) Explain, in terms of black and white balls selected from an urn, why

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$$

(3 points) (c) A police department consists of 10 officers. If the policy is to have 5 officers on patrol in the streets, 2 full time at the station and 3 on reserve at the station then how many divisions of the 10 officers into the three groups are possible?

(2 points) (d) How many different ways can 22 people divide themselves up to play soccer (11 per side).

(10 points) 2. The US Bureau of census collected the following data

Region	% of US Population	% seniors
Northeast	19.0	13.8
Midwest	23.1	13.0
South	35.5	12.8
West	22.4	11.1

For instance, 13.8% of Northeast residents are seniors. What percentage of seniors are Northeast residents?

(10 points) 3. Ten numbers are rounded to the nearest integer and then summed. Using the central limit theorem, determine the probability that the sum of the rounded numbers will equal the rounded sum of the unrounded numbers. You may assume that the roundoffs for the ten numbers are independent and uniformly distributed in $(-0.5, 0.5)$.

4. Earthquakes happen according to a Poisson process with rate λ but each quake is detected with probability p, independently. Let X denote the number of earthquakes in a fixed unit time interval and let X_c denote the number that are detected in the same time interval.

(2 points) (a) Write a formula for the moment generating function of a discrete random variable Y in terms of the probability mass distribution of Y.

(2 points) (b) What is the moment generating function of X. Explain, using part (a).

(1 points) (c) What is the distribution of X_c conditioned on $X = n$?

(2 points) (d) Calculate $E(e^{tX_c}|X = n)$.

(2 points) (e) What is the moment generating function of X_c?

(1 points) (f) What is the distribution of X_c?

5. Let d be a positive integer. Let $\vec{e}_i$ be the vector with d components whose i^{th} coordinate is one and whose other coordinates are zero. Consider the simple random walk on $\mathbb{Z}^d$ starting from $\vec{0} \in \mathbb{Z}^d$. The position of the walk after n steps is $\vec{S}_n = \vec{X}_1 + \ldots + \vec{X}_n$, where the $\vec{X}_i$ are i.i.d. and take the values $\pm\vec{e}_1, \ldots, \pm\vec{e}_d$ with equal probabilities $\frac{1}{2d}$.

(4 points) (a) Compute the characteristic function $\phi_1(\vec{k})$ of $\vec{X}_1$.

(2 points) (b) Expand $\phi_1(\vec{k})$ in a Taylor expansion about $\vec{k} = \vec{0}$, to second order.

(4 points) (c) Prove that the random walk is recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$. You may use fact that the expected number of visits to $\vec{0}$ is given by the integral

$$\int_{[-\pi,\pi]^d} \frac{1}{1 - \phi_1(\vec{k})} \frac{d^d\vec{k}}{(2\pi)^d}.$$

6. Consider the Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c|cccccc|} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 2 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ 3 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 4 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 5 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

(3 points) (a) Draw the transition diagram showing the six states with arrows indicating possible transitions and their probabilities.

(4 points) (b) Determine all the irreducibility classes of this Markov chain.

(3 points) (c) Determine which states are recurrent and which are transient.

7. This problem considers a modification of the Ehrenfest chain. Let M be a positive integer. Suppose that M molecules are distributed among two urns. We choose a molecule at random and remove it from its urn, and then choose an urn at random and place the removed molecule into the chosen urn. Let X_n denote the number of molecules in urn number one, after the n^{th} step. This defines a Markov chain.

(4 points) (a) Write formulas for the transition matrix elements P_{ij}.

(3 points) (b) Guess the stationary distribution for general M.

(3 points) (c) Verify that your guess in (c) is correct.

8. Fix a number p with $0 < p < 1$. Consider the Markov chain on the non-negative integers $\{0, 1, 2, \ldots\}$ whose transition probabilities are given by

$$P_{n,n+1} = p, \quad P_{n,0} = 1 - p, \quad \text{for all } n \geq 0.$$

(2 points) (a) Is this Markov chain irreducible? Explain.

(2 points) (b) Is the state 0 periodic or aperiodic? Explain.

(3 points) (c) Suppose that the Markov chain is initially in state 0, and let T_0 denote the time of first return to 0 (i.e., T_0 is the smallest value of $n > 0$ such that $X_n = 0$, if such a value exists, and otherwise $T_0 = \infty$). Determine the probability mass function of T_0.

(1 points) (d) What exactly does it mean to say that 0 is a recurrent state?

(2 points) (e) Prove that 0 a recurrent state.

Table 1: Mean and Variances

Distribution	Mean	Variance
Bin (n, p)	np	$np(1-p)$
Geometric (p)	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson (λ)	λ	λ
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exp (λ)	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Table 2: cdf of normal distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990