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The University of British Columbia

MATH 317, Section 101

Final Exam – December 2016

Family Name ____________________ **Given Name** ____________________

Student Number ____________________ **Signature** ____________________

No notes or calculator allowed

Question	Points	Score
1	10	
2	10	
3	6	
4	6	
5	8	
6	8	
7	10	
8	12	
9	12	
10	18	
Total:	100	

1. A curve in $\mathbb{R}^3$ is given by $\vec{\mathbf{r}}(t) = (t^2,\ t,\ t^3)$.

(5 pt) (a) Find the parametric equations of the tangent line to the curve at the point $P(1, -1, -1)$.

(5 pt) (b) Find an equation for the osculating plane of the curve at the point $Q(1, 1, 1)$.

2. A curve in $\mathbb{R}^3$ is given by $\vec{\mathbf{r}}\,(t) = (\sin t - t\cos t)\,\hat{\mathbf{\imath}} + (\cos t + t\sin t)\,\hat{\mathbf{\jmath}} + t^2\,\hat{\mathbf{k}}, \quad 0 \le t < \infty.$

(6 pt) (a) Find the length of the curve $\vec{\mathbf{r}}\,(t)$ from $\vec{\mathbf{r}}\,(0) = (0, 1, 0)$ to $\vec{\mathbf{r}}\,(\pi) = (\pi, -1, \pi^2)$.

(4 pt) (b) Find the curvature of the curve at time $t > 0$.

(6 pt) 3. Let $\vec{\mathbf{F}}(x, y, z) = e^x \sin y\,\hat{\mathbf{\imath}} + [ae^x \cos y + bz]\,\hat{\mathbf{\jmath}} + cx\,\hat{\mathbf{k}}$. For which values of the constants a, b, c is $\int_C \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{r}} = 0$ for all closed paths C?

4. Let $\vec{\mathbf{F}}(x, y) = P\,\hat{\mathbf{\imath}} + Q\,\hat{\mathbf{\jmath}}$ be a smooth plane vector field defined for $(x, y) \neq (0, 0)$, and suppose $Q_x = P_y$ for $(x, y) \neq (0, 0)$. In the following $I_j = \int_{C_j} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for integer j, and all C_j are positively oriented circles. Suppose $I_1 = \pi$ where C_1 is the circle $x^2 + y^2 = 1$.

(2 pt) (a) Find I_2 for $C_2 : (x - 2)^2 + y^2 = 1$. Explain briefly.

(2 pt) (b) Find I_3 for $C_3 : (x - 2)^2 + y^2 = 9$. Explain briefly.

(2 pt) (c) Find I_4 for $C_4 : (x - 2)^2 + (y - 2)^2 = 9$. Explain briefly.

(8 pt) 5. Evaluate the line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$ where $\vec{\mathbf{F}}(x,y) = xy^2\,\hat{\mathbf{\imath}} + ye^x\,\hat{\mathbf{\jmath}}$, C is the boundary of the rectangle R: $0 \le x \le 3$, $-1 \le y \le 1$, and $\vec{\mathbf{n}}$ is the unit outer normal vector of C.

(8 pt) 6. Find the work done by the force field $\vec{\mathbf{F}}(x, y, z) = \left(x - y^2, y - z^2, z - x^2\right)$ on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.

(10 pt) 7. Evaluate the surface integral $\iint_S z^2 dS$ where S is the part of the cone $x^2 + y^2 = 4z^2$ where $0 \leq x \leq y$ and $0 \leq z \leq 1$.

(12 pt) 8. Let $\vec{\mathbf{F}}$ be the vector field defined by

$$\vec{\mathbf{F}}(x,y,z) = (y^3 z + 2x)\,\hat{\mathbf{\imath}} + (3y - e^{\sin z})\,\hat{\mathbf{\jmath}} + (e^{x^2+y^2} + z)\,\hat{\mathbf{k}}.$$

Calculate the flux integral $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where S is the boundary surface of the solid region

$$E: \quad 0 \le x \le 2, \quad 0 \le y \le 2, \quad 0 \le z \le 2 + y$$

with outer normal.

9. Consider the vector field $\vec{\mathbf{F}}(x, y, z) = z^2\,\hat{\mathbf{\imath}} + x^2\,\hat{\mathbf{\jmath}} + y^2\,\hat{\mathbf{k}}$ in $\mathbb{R}^3$.

(6 pt) (a) Compute the line integral $I_1 = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C_1 is the curve consisting of three line segments, L_1 from $(2, 0, 0)$ to $(0, 2, 0)$, then L_2 from $(0, 2, 0)$ to $(0, 0, 2)$, finally L_3 from $(0, 0, 2)$ to $(2, 0, 0)$.

(6 pt) (b) A simple closed curve C_2 lies on the plane $E : x + y + z = 2$, enclosing a region R on the plane of area 3, and oriented in a counterclockwise direction as observed from the positive x-axis. Compute the line integral $I_2 = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

(18 pt) 10. Circle True or False for each of the following statement. No justifications necessary.

Recall that $f \in C^k$ means that all derivatives of f up to order k exist and are continuous.

(a) True or False? $\nabla \times (f\nabla f) = 0$ for all C^2 scalar function f in $\mathbb{R}^3$.

(b) True or False? $\operatorname{div}(f\vec{\mathbf{F}}) = \nabla f \cdot \vec{\mathbf{F}} + f \operatorname{div} \vec{\mathbf{F}}$ for all C^1 scalar function f and C^1 vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$.

(c) True or False? A smooth space curve C with constant curvature $\kappa = 0$ must be a straight line.

(d) True or False? A smooth space curve C with constant curvature $\kappa \neq 0$ must be part of a circle of radius $1/\kappa_0$.

(e) True or False? If f is any smooth function defined in $\mathbb{R}^3$ and if C is any circle, then $\int_C \nabla f \cdot d\vec{\mathbf{r}} = 0$.

(f) True or False? Suppose $\vec{\mathbf{F}}$ is a smooth vector field in the region D which is $\mathbb{R}^3$ without the origin. If $\operatorname{div} \vec{\mathbf{F}} > 0$ through out D, then flux of $\vec{\mathbf{F}}$ through the sphere of radius 5 with center at the origin is positive.

(g) True or False? If $\vec{\mathbf{F}}$ is a smooth vector field in $\mathbb{R}^3$ and $\operatorname{div} \vec{\mathbf{F}} = 0$ everywhere, then, for every sphere, the flux *out of* one hemisphere is equal to the flux *into* the opposite hemisphere.

(h) True or False? Let $\vec{\mathbf{F}}(x, y, z)$ be a continuously differentiable vector field which is defined for every (x, y, z). Then, $\iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 0$ for any closed surface S.
(A *closed surface* is a surface that is the boundary of a solid region.)

(i) True or False? The parametrizations

$$\vec{\mathbf{r}}_1(t) = (t, t^2), \qquad -\infty < t < \infty$$

and

$$\vec{\mathbf{r}}_2(t) = (t^2, t^4), \qquad -\infty < t < \infty$$

describe exactly the same plane curve.

Math 317 Calculus IV Formulas for Final Exam

1. For a curve $\vec{\mathbf{r}}(t)$, $s = \displaystyle\int_0^t |\vec{\mathbf{r}}'(\tau)|d\tau$, $\dfrac{ds}{dt} = |\vec{\mathbf{r}}'|$, $ds = |\vec{\mathbf{r}}'(t)|dt$
2. $\mathbf{T} = \dfrac{\vec{\mathbf{r}}'}{|\vec{\mathbf{r}}'|}$, $\mathbf{N} = \dfrac{\mathbf{T}'}{|\mathbf{T}'|}$, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
3. $\kappa = |\dfrac{d\mathbf{T}}{ds}| = \dfrac{|\mathbf{T}'|}{|\vec{\mathbf{r}}'|} = \dfrac{|\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''|}{|\vec{\mathbf{r}}'|^3}$, $\kappa\mathbf{N} = \dfrac{d\mathbf{T}}{ds}$
4. For $y = f(x)$, $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$
5. Green's theorem: $\int_C P dx + Q dy = \iint_D (Q_x - P_y)\, dA$
6. For a surface S given by $\vec{\mathbf{r}}(u, v) : D \to \mathbb{R}^3$, the surface area is $\int_S dS = \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dudv$
   $\int_S \rho\, dS = \iint_D \rho(\vec{\mathbf{r}})\, |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, du\, dv$
   $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, du\, dv$
7. For a graph S given by $z = f(x, y)$, $(x, y) \in D$, the surface area is
   $\int_S dS = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dxdy$
   $\int_S \rho\, dS = \iint_D \rho(\vec{\mathbf{r}}) \sqrt{1 + f_x^2 + f_y^2}\, dxdy$
   $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}) \cdot (-f_x, -f_y, 1)\, dx\, dy$
8. For a surface of revolution $S : r = f(z)$, $a \le z \le b$, the surface area is
   $\int_S dS = \int_a^b 2\pi f(z)\sqrt{1 + [f'(z)]^2}\, dz$
9. Stokes theorem: For a surface S with boundary curve C, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$.
10. Divergence theorem: For a solid region E with boundary surface S,
    $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV$.