*This final exam has **6 questions** on **12 pages**, for a total of 60 points.*

Duration: 3 hours

- Write your name or your student number on **every** page.
- **You need to show enough work to justify your answers.**
- Continue on the **back of the previous page** if you run out of space. You also have extra space at the end of the booklet.
- This is a closed-book examination. **None of the following are allowed**: documents or electronic devices of any kind (including calculators, cell phones, etc.)

LAST name: ______________________________

First name: (including all middle names): ______________________________

Student Number: ______________________________

Signature: ______________________________

Circle the name of your instructor: Rachel Ollivier Justin Tzou

Question:	1	2	3	4	5	6	Total
Points:	10	8	10	10	10	12	60
Score:							

We recall that for a vector field $\mathbf{F}$ in $\mathbb{R}^3$, we have:

$$\mathbf{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$$

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}\,.$$

The unit tangent vector $\mathbf{T}(t)$, principal unit normal vector $\mathbf{N}(t)$, binormal vector $\mathbf{B}(t)$, and curvature $\kappa(t)$, of a curve in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t)$ are given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\,, \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}\,, \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}\,.$$

The volume of a sphere with radius a is $\frac{4}{3}\pi a^3$.

1. Consider the closed region enclosed by the curves $y = x^2 + 4x + 4$ and $y = 4 - x^2$. Let C be its boundary and suppose that C is oriented counter-clockwise.

2 marks (a) Draw the **oriented** curve C carefully in the $x - y$-plane.

8 marks (b) Determine the value of

$$\oint_C xy\,dx + (e^y + x^2)\,dy.$$

Hint: do not compute the integral directly.

2. Consider the vector field $\mathbf{F}(x, y, z) = \langle \cos x, 2 + \sin y, e^z \rangle$.

1 mark (a) Compute the curl of $\mathbf{F}$.

1 mark (b) Is there a function f such that $\mathbf{F} = \nabla f$? Justify your answer.

6 marks (c) Compute the integral of $\mathbf{F}$ along the curve C parametrized by $\mathbf{r}(t) = \langle t, \cos t, \sin t \rangle$ with $0 \leq t \leq 3\pi$.

3. Let S be the sphere of radius 3, centered at the origin and with outward orientation. Given the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, x + z \rangle$:

7 marks

(a) Calculate (using the definition) the flux of $\mathbf{F}$ through S

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

That is, compute the flux by evaluating the surface integral directly.
Hint: if you give a parametrization $\mathbf{r}(\theta, \phi)$ *of the sphere using the usual* θ, ϕ *of the spherical coordinates, then* $\mathbf{r}_\theta \times \mathbf{r}_\phi$ *and* $\mathbf{r}_\phi \times \mathbf{r}_\theta$ *both give a vector of the form* $\alpha(\phi)\mathbf{r}(\theta, \phi)$ *for some function* $\alpha(\phi)$*. Determining which one to use is important in the calculation above. Here,* ϕ *is the angle measured from the positive* z*-axis.*

3 marks

(b) Calculate the same flux using the divergence theorem.

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4. We consider the cone with equation $z = \sqrt{x^2 + y^2}$. Note that its tip, or vertex, is located at the origin $(0, 0, 0)$. The cone is oriented in such a way that the normal vectors point downwards (and away from the z axis). In the parts below, both S_1 and S_2 are oriented this way.

   Let $\mathbf{F} = \langle -zy, zx, xy\cos(yz)\rangle$.

5 marks

   (a) Let S_1 be the part of the cone that lies between the planes $z = 0$ and $z = 4$. Note that S_1 does not include any part of the plane $z = 4$. Use Stokes' theorem to determine the value of
$$\iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S}\,.$$
   Make a sketch indicating the orientations of S_1 and of the contour(s) of integration.

5 marks

   (b) Let S_2 be the part of the cone that lies below the plane $z = 4$ and above $z = 1$. Note that S_2 does not include any part of the planes $z = 1$ and $z = 4$. Determine the flux of $\nabla \times \mathbf{F}$ across S_2. Justify your answer, including a sketch indicating the orientations of S_2 and of the contour(s) of integration.

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10 marks 5. Consider the cube of side length 1 that lies entirely in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with one corner at the origin and another corner at point $(1, 1, 1)$. As such, one face lies in the plane $x = 0$, one lies in the plane $y = 0$, and another lies in the plane $z = 0$. The other three faces lie in the planes $x = 1$, $y = 1$, and $z = 1$.

Denote S as the **open** surface that consists of the union of the 5 faces of the cube **that do not lie in the plane** $z = 0$. The surface S is oriented in such a way that the unit normal vectors point outwards (that is, the orientation of S is such that the unit normal vectors on the top face point towards positive z-directions). Determine the value of

$$I = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}$ is the vector field given by

$$\mathbf{F} = \langle y\cos(y^2) + z - 1,\ \frac{z}{x+1} + 1\,,\ xye^{z^2} \rangle\,.$$

Hint: do not compute the integral directly. .

6. Consider the curve C in 3 dimensions given by

$$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + \sqrt{3}t^2\mathbf{k}$$

for $t \in \mathbb{R}$.

1 mark (a) Compute the unit tangent vector $\mathbf{T}(t)$.
It will be a vector of the form $\mathbf{T}(t) = \dfrac{\langle 1, at, bt\rangle}{\sqrt{1+4t^2}}$ *where* a *and* b *are nonzero constant real numbers.*

1 mark (b) Compute the unit normal vector $\mathbf{N}(t)$.
It will be a vector of the form $\mathbf{N}(t) = \dfrac{\langle -4t, \alpha, \beta\rangle}{2\sqrt{1+4t^2}}$ *where* α *and* β *are nonzero constant real numbers.*

1 mark (c) Show that the binormal vector $\mathbf{B}$ to this curve does not depend on t and is one of the following vectors:

$$① \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \\ 0 \end{pmatrix} \quad ② \begin{pmatrix} 0 \\ \sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad ③ \begin{pmatrix} 0 \\ -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad ④ \begin{pmatrix} 0 \\ -1/2 \\ \sqrt{3}/2 \end{pmatrix}.$$

This implies that C is a plane curve.

2 marks (d) According to your choice of vector ① ② ③ or ④, give the equation of the plane containing C. *You can get credit for this question even if your choice of vector is wrong.*

1 mark (e) Compute the curvature $\kappa(t)$ of the curve.
It will be a function of the form $\kappa(t) = \dfrac{\gamma}{(1+4t^2)^{3/2}}$, *where* γ *is a positive constant real number.*

1 mark (f) Are there point(s) where the curvature is maximal? If yes, give the coordinates of the point(s). If no, justify your answer.

1 mark (g) Are there point(s) where the curvature is minimal? If yes, give the coordinates of the point(s). If no, justify your answer.

4 marks (h) Let

$$\mathbf{u} := 2\mathbf{i}, \quad \mathbf{v} := \mathbf{j} + \sqrt{3}\mathbf{k}, \quad \mathbf{w} := -\sqrt{3}\mathbf{j} + \mathbf{k}.$$

i) Express $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in terms of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

ii) Using i), write $\mathbf{r}(t)$ in the form

$$a(t)\mathbf{u} + b(t)\mathbf{v} + c(t)\mathbf{w}$$

where $a(t)$, $b(t)$ and $c(t)$ are functions you have to determine. *You should find that one of these functions is zero.*

iii) Draw the curve given by $\langle a(t), b(t)\rangle$ in the $x-y$-plane.

iv) Is the drawing consistent with parts (f) and (g)? Explain.

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