Final Exam

December 12, 2012
8:30–11:00

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (5 points)

(a) Consider the parametrized space curve

$$\vec{r}(t) = \langle \cos(t), \sin(t), t^2 \rangle \,.$$

Find a parametric form for the tangent line at the point corresponding to $t = \pi$.

(b) Find the tangential component $a_T(t)$ of acceleration, as a function of t, for the parametrized space curve of (a).

Problem 2. (5 points)

(a) Let

$$\vec{r}(t) = \langle 2\sin^3 t, 2\cos^3 t, 3\sin t \cos t \rangle \,.$$

Find the unit tangent vector to this parametrized curve at $t = \pi/3$, pointing in the direction of increasing t.

(b) Reparametrize the vector function $\vec{r}(t)$ from (a) with respect to arc length measured from the point $t = 0$ in the direction of increasing t.

Problem 3. (6 points)

(a) Consider the vector field $\vec{F} = \langle 3y, x - 1 \rangle$ in $\mathbb{R}^2$. Compute the line integral

$$\int_L \vec{F} \cdot d\vec{r} \,,$$

where L is the line segment from $(1, 1)$ to $(2, 2)$.

(b) Find an oriented path C from $(2, 2)$ to $(1, 1)$ such that

$$\int_C \vec{F} \cdot d\vec{r} = 4 \,,$$

where $\vec{F}$ is the vector field from (a).

Problem 4. (6 points)

(a) Find the curl of the vector field $\vec{F} = \langle 2 + x^2 + z, 0, 3 + x^2 z\rangle$.

(b) Let C be the curve in $\mathbb{R}^3$ from the point $(0,0,0)$ to the point $(2,0,0)$, consisting of three consecutivce line segments connecting the points $(0,0,0)$ to $(0,0,3)$, $(0,0,3)$ to $(0,1,0)$, and $(0,1,0)$ to $(2,0,0)$. Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}\,,$$

where $\vec{F}$ is the vector field from (a).

Problem 5. (6 points)

(a) Consider the surface S given by the equation

$$x^2 + z^2 = \cos^2 y\,,$$

Find an equation for the normal plane to S at the point $(\frac{1}{2}, \frac{\pi}{4}, \frac{1}{2})$.

(b) Compute the integral

$$\iint_S \sin y\, dS\,,$$

where S is the part of the surface from (a) lying between the planes $y = 0$ and $y = \frac{1}{2}\pi$.

Problem 6. (6 points)

(a) Let S be the bucket shaped surface consisting of the cylindrical surface $y^2 + z^2 = 9$ between $x = 0$ and $x = 5$, and the disc inside the yz-plane of radius 3 centered at the origin. (The bucket S has a bottom, but no lid.) Orient S in such a way that the unit normal points outward. Compute the flux of the vector field $\nabla \times \vec{G}$ through S, where $\vec{G} = \langle x, -z, y\rangle$.

(b) Compute the flux of the vector field $\vec{F} = \langle 2 + z, xz^2, x\cos y\rangle$ through S, where S is as in (a).

Problem 7. (6 points)

(a) Find the divergence of the vector field $\vec{F} = \langle z + \sin y, zy, \sin x \cos y\rangle$.

(b) Find the flux of the vector field $\vec{F}$ of (a) through the sphere of radius 3 centred at the origin in $\mathbb{R}^3$.

Problem 8. (10 points)
True or false? Put the answers in your exam booklet, please. No justifictions necessary.

1. $\vec{\nabla} \times (\vec{a} \times \vec{r}) = \vec{0}$, here $\vec{a}$ is a constant vector in $\mathbb{R}^3$, and $\vec{r}$ is the vector field $\vec{r} = \langle x, y, z \rangle$.

2. $\vec{\nabla} \cdot (\vec{\nabla} f) = 0$, for all scalar fields f on $\mathbb{R}^3$ with continuous second partial derivatives.

3. $\vec{\nabla}(\vec{\nabla} \cdot \vec{F}) = \vec{0}$, for every vector field $\vec{F}$ in $\mathbb{R}^3$ with continuous second partial derivatives.

4. Suppose $\vec{F}$ is a vector field with continuous partial derivatives in the region D, where D is $\mathbb{R}^3$ without the origin. If $\operatorname{div} \vec{F} = 0$, then the flux of $\vec{F}$ through the sphere of radius 5 with center at the origin is 0.

5. Suppose $\vec{F}$ is a vector field with continuous partial derivatives in the region D, where D is $\mathbb{R}^3$ without the origin. If $\vec{\nabla} \times \vec{F} = 0$, then $\int_C \vec{F} \cdot d\vec{r} = 0$, for every simple and smooth closed curve C in $\mathbb{R}^3$ which avoids the origin.

6. If a vector field $\vec{F}$ is defined and has continuous partial derivatives everywhere in $\mathbb{R}^3$, and it satisfies $\operatorname{div} \vec{F} > 0$, everywhere, then, for every sphere, the flux *out of* one hemisphere is larger than the flux *into* the opposite hemisphere.

7. If $\vec{r}(t)$ is a path in $\mathbb{R}^3$ with constant curvature κ, then $\vec{r}(t)$ parametrizes part of a circle of radius $1/\kappa$.

8. The vector field $\vec{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z \rangle$ is conservative in its domain, which is $\mathbb{R}^3$ without the z-axis.

9. If all flow lines of a vector field in $\mathbb{R}^3$ are parallel to the z-axis, then the circulation of the vector field around every closed curve is 0.

10. If the speed of a moving particle is constant, then its acceleration is orthogonal to its velocity.