The University of British Columbia

Final Examination - December 16, 2010

Mathematics 317

Instructor: Katherine Stange

Closed book examination | Time: 3 hours

Name ______________________ **Signature** ______________________

Student Number ____________________

Special Instructions:

- Be sure that this examination has 17 pages. Write your name on top of each page. Do not detach any pages.
- No calculators or notes are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

1		7
2		9
3		8
4		8
5		14
6		8
7		7
8		7
9		22
10		10
Total		100

Problem 1 of 10 [7 points]

Let $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ be the position vector of a particle as a function of time $t \geq 0$.

(a) (2 points) Find the velocity of the particle as a function of time t.

(b) (5 points) Find the arclength of its path between $t = 1$ and $t = 2$.

Problem 2 of 10 [9 points]

Let C be the upper half of the unit circle centred on $(1, 0)$ (i.e. that part of the circle which lies above the x axis), oriented clockwise. Compute the line integral

$$\int_C xy\, dy.$$

Problem 3 of 10 [8 points]

Let S be the surface given by

$$\mathbf{s}(u,v) = \langle u+v, u^2+v^2, u-v \rangle, \quad -2 \le u \le 2, -2 \le v \le 2$$

(a) (4 points) Find the tangent plane to the surface at the point $(2,2,0)$.

(b) (2 points) This is a surface you are familiar with. What surface is it (it may be just a portion of one of the following)? Circle one:

sphere helicoid ellipsoid saddle parabolic bowl cylinder cone plane

(c) (2 points) In which direction does the parametrisation orient the surface?
Circle the correct choices: In the (positive / negative) (x / y / z) direction.

Problem 4 of 10 [8 points]

Let

$$\mathbf{F}(x,y) = \langle y^2 - e^{-y^2} + \sin x,\ 2xye^{-y^2} + x\rangle.$$

Let C be the boundary of the triangle with vertices $(0,0)$, $(1,0)$ and $(1,2)$, oriented counter-clockwise. Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Hint: Don't do the integral directly.

Problem 5 of 10 [14 points]

Let

$$\mathbf{F}(x, y, z) = \langle \frac{y}{x} + x^{1+x^2},\ x^2 - y^{1+y^2},\ \cos^5(\ln z)\rangle.$$

(a) (2 points) Write down the domain D of $\mathbf{F}$.

(b) (1 points) Circle the correct statement(s):

(a) D is connected.

(b) D is simply connected.

(c) D is disconnected.

(c) (2 points) Compute $\nabla \times \mathbf{F}$.

(d) (8 points) Let C be the square with corners $(3 \pm 1, 3 \pm 1)$ in the plane $z = 2$, oriented clockwise (viewed from above, i.e. down z-axis). Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Hint: Don't do it directly.

(e) (1 point) Is $\mathbf{F}$ conservative? Circle one: Yes No

Problem 6 of 10 [8 points]

Let

$$\mathbf{F}(x,y,z) = \langle e^{-y^2} + y^{1+x^2} + \cos(z),\ -z,\ y\rangle.$$

Let S be the surface which consists of two parts:

- the portion of the paraboloid $y^2 + z^2 = 4(x+1)$ satisfying $0 \le x \le 3$ and
- the portion of the sphere $x^2 + y^2 + z^2 = 4$ satisfying $x < 0$.

and is oriented outward. Compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Hint: Don't do it directly.

Problem 7 of 10 [7 points]

Let

$$\mathbf{F}(x, y, z) = \langle 1 + z^{1+z^{1+z}}, 1 + z^{1+z^{1+z}}, 1 \rangle.$$

Let S be the portion of the surface

$$x^2 + y^2 = 1 - z^4$$

which is above the xy-plane. What is the flux of $\mathbf{F}$ downward through S?

Problem 8 of 10 [7 points]

Let

$$\mathbf{F}(x,y) = \langle 1,\ y\,g(y) \rangle,$$

and suppose that $g(y)$ is a function defined everywhere with everywhere continuous partials. Show that for any curve C whose endpoints P and Q lie on the x-axis,

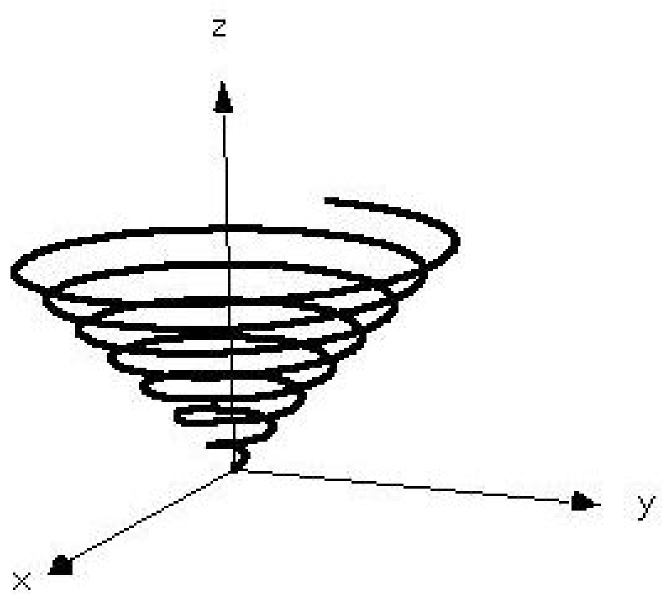
$$\text{distance between } P \text{ and } Q \;=\; \left| \int_C \mathbf{F} \cdot d\mathbf{r} \right|.$$

Problem 9 of 10 [22 points]

Short Answer. Only your answer counts.

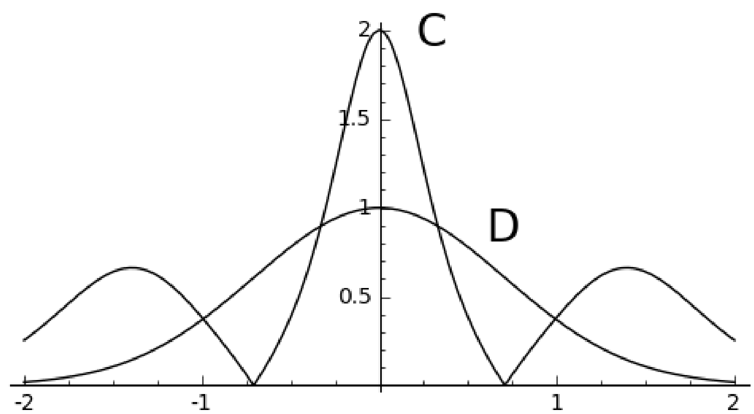
(a) (1 point) In the curve shown below (a helix lying in the surface of a cone), is the curvature increasing, decreasing, or constant as z increases?

Circle the answer: increasing decreasing constant



(b) (2 points) Of the two functions shown below, one is a function $f(x)$ and one is its curvature $\kappa(x)$. Which is which?

$f(x)$ is (circle one): C D



(c) (3 points) Let C be the curve of intersection of the cylinder $x^2 + z^2 = 1$ and the saddle $xz = y$. Parametrise C.

$\mathbf{r}(t) =$

domain:

(d) (4 points) Let H be the helical ramp (also known as a helicoid) which revolves around the z-axis in a clockwise direction viewed from above, beginning at the y-axis when $z = 0$, and rising 2π units each time it makes a full revolution. Let S be the the portion of H which lies outside the cylinder $x^2 + y^2 = 4$, above the $z = 0$ plane and below the $z = 5$ plane.

Choose one of the following functions and give the domain on which the function you have chosen parametrises S. (Hint: Only one of the following functions is possible.)

(a) $s(u, v) = \langle u \cos v, u \sin v, u\rangle$
(b) $s(u, v) = \langle u \cos v, u \sin v, v\rangle$
(c) $s(u, v) = \langle u \sin v, u \cos v, u\rangle$
(d) $s(u, v) = \langle u \sin v, u \cos v, v\rangle$

domain:

(e) (2 points) Write down a parametrised curve of zero curvature and arclength 1.

$\mathbf{r}(t) =$

domain:

(f) (2 points) If $\nabla \cdot \mathbf{F}$ is a constant C on all of $\mathbb{R}^3$, and S is a cube of unit volume such that the flux outward through each side of S is 1, what is C?

$C =$

(g) (2 points) Let

$$\mathbf{F}(x, y) = \langle ax + by, cx + dy \rangle.$$

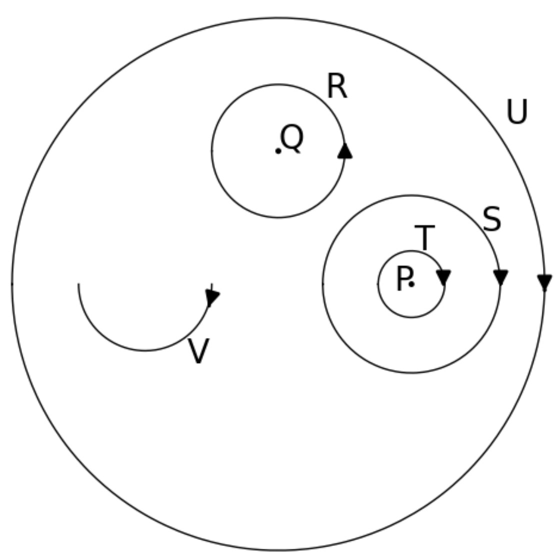
Give the full set of a, b, c and d such that $\mathbf{F}$ is conservative.

(h) (1 point) If $\mathbf{r}(s)$ has been parametrised by arclength (i.e. s is arclength), what is the arclength of $\mathbf{r}(s)$ between $s = 3$ and $s = 5$?

arclength:

(i) (2 points) Let $\mathbf{F}$ be a 2D vector field which is defined everywhere *except* at the points marked P and Q. Suppose that $\nabla \times \mathbf{F} = 0$ everywhere on the domain of $\mathbf{F}$. Consider the five curves R, S, T, U, and V shown in the picture. Which of the following is *necessarily* true?

(a) $\int_S \mathbf{F} \cdot d\mathbf{r} = \int_T \mathbf{F} \cdot d\mathbf{r}$
(b) $\int_R \mathbf{F} \cdot d\mathbf{r} = \int_S \mathbf{F} \cdot d\mathbf{r} = \int_T \mathbf{F} \cdot d\mathbf{r} = \int_U \mathbf{F} \cdot d\mathbf{r} = 0$
(c) $\int_R \mathbf{F} \cdot d\mathbf{r} + \int_S \mathbf{F} \cdot d\mathbf{r} + \int_T \mathbf{F} \cdot d\mathbf{r} = \int_U \mathbf{F} \cdot d\mathbf{r}$
(d) $\int_U \mathbf{F} \cdot d\mathbf{r} = \int_R \mathbf{F} \cdot d\mathbf{r} + \int_S \mathbf{F} \cdot d\mathbf{r}$
(e) $\int_V \mathbf{F} \cdot d\mathbf{r} = 0$

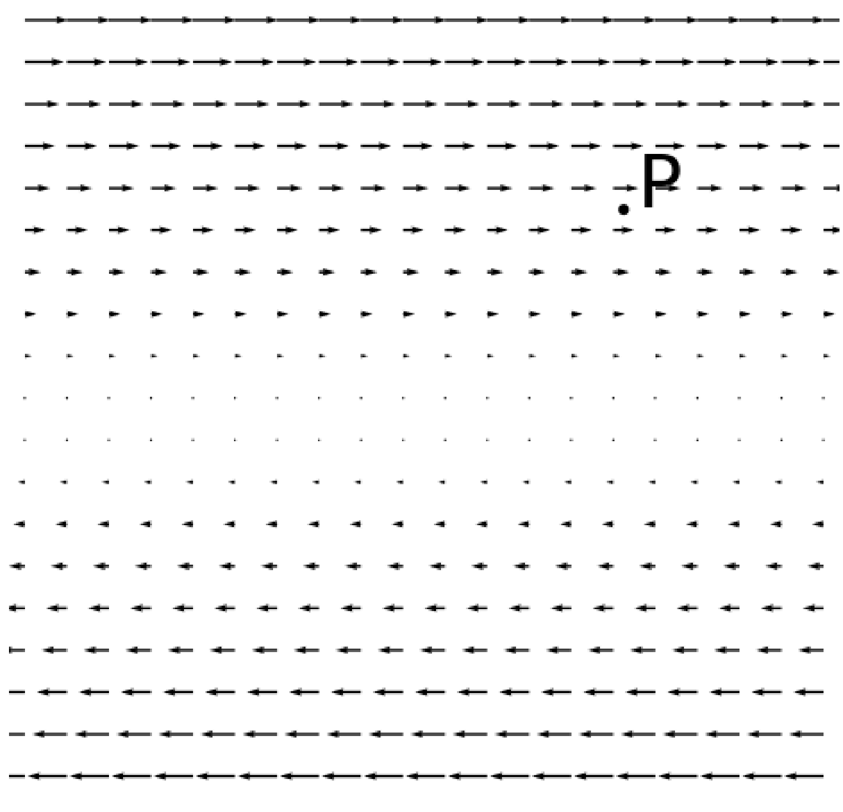


(j) (2 points) Write down a 3D vector field $\mathbf{F}$ such that for all closed surfaces S, the volume enclosed by S is equal to

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

$$\mathbf{F}(x, y, z) =$$

(k) (1 point) Consider the vector field $\mathbf{F}$ shown below.
curl $\mathbf{F}$ at P is (circle one): positive negative zero



Problem 10 of 10 [10 points]

Say whether the following statements are true (**T**) or false (**F**). Only your answer counts. **You will get 1 point for each correct answer, 0 for each wrong or unanswered.**

T F If $\mathbf{F}$ is a 3D vector field defined on all of $\mathbb{R}^3$, and S_1 and S_2 are two surfaces with the same boundary, but $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} \neq \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, then $\operatorname{div} \mathbf{F}$ is not zero anywhere.

T F If $\mathbf{F}$ is a vector field satisfying $\operatorname{curl} \mathbf{F} = 0$ whose domain is not simply-connected, then $\mathbf{F}$ is not conservative.

T F The osculating circle of a curve C at a point has the same unit tangent vector, unit normal vector, and curvature as C at that point.

T F A planet orbiting a sun has period proportional to the cube of the major axis of the orbit.

T F For any 3D vector field $\mathbf{F}$, $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

T F A field whose divergence is zero everywhere in its domain has $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for all closed surfaces S in its domain.

T F The gravitational force field is conservative.

T F If $\mathbf{F}$ is a field defined on all of $\mathbf{R}^3$ such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 3$ for some curve C, then $\nabla \times \mathbf{F}$ is non-zero at some point.

T F The normal component of acceleration for a curve of constant curvature is constant.

T F The curve defined by

$$\mathbf{r}_1(t) = \cos(t^4)\,\mathbf{i} + 3t^4\,\mathbf{j}, \quad -\infty < t < \infty,$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{k}, \quad -\infty < t < \infty.$$

[blank page - do NOT detach this page from the exam]

[blank page - do NOT detach this page from the exam]