The University of British Columbia
Final Examination - April 26, 2008

Mathematics 317 Section 202
Instructors: Jim Bryan and Hendryk Pfeiffer

Closed book examination. Time: 3 hours

Name ______________________________ **Signature** ______________________________

Student Number ____________________

Special Instructions:

- Be sure that this examination has 15 pages. Write your name on top of each page.
- One single sided letter sized formula sheet is allowed. No further notes and no calculators are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBC-card for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  - (b) Speaking or communicating with other candidates.
  - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		8
4		12
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Problem 1 of 10 [10 points]

Consider the curve C given by

$$\mathbf{r}(t) = \frac{1}{3}t^3\,\mathbf{i} + \frac{1}{\sqrt{2}}t^2\,\mathbf{j} + t\,\mathbf{k}, \qquad -\infty < t < \infty.$$

(1) [3 points] Find the unit tangent $\mathbf{T}(t)$ as a function of t.

(2) [4 points] Find the curvature $\kappa(t)$ as a function of t.

(3) [3 points] Determine the principal normal vector $\mathbf{N}$ at the point $(\frac{8}{3}, 2\sqrt{2}, 2)$.

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Problem 2 of 10 [10 points]

A particle of mass $m = 1$ has position $\mathbf{r}_0 = \frac{1}{2}\,\mathbf{k}$ and velocity $\mathbf{v}_0 = \frac{\pi^2}{2}\,\mathbf{i}$ at time $t = 0$. It moves under a force

$$\mathbf{F}(t) = -3t\,\mathbf{i} + \sin t\,\mathbf{j} + 2e^{2t}\,\mathbf{k}.$$

(1) [4 points] Determine the position $\mathbf{r}(t)$ of the particle depending on t.

(2) [3 points] At what time after time $t = 0$ does the particle cross the plane $x = 0$ for the first time?

(3) [3 points] What is the velocity of the particle when it crosses the plane $x = 0$ in part (2)?

Problem 3 of 10 [8 points]

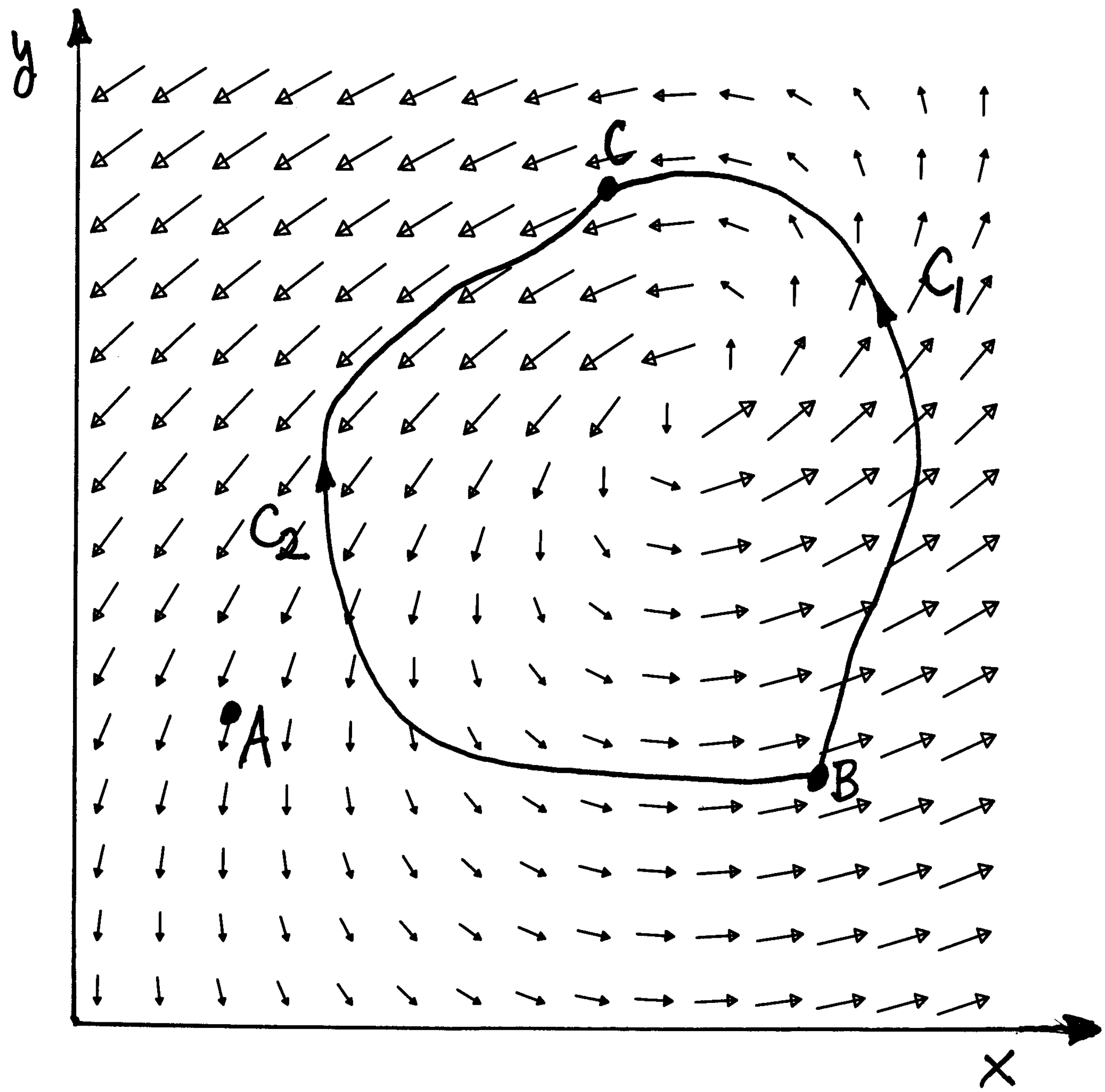
On the following page, the vector field

$$\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

is plotted. In the following questions, give the answer that is best supported by the plot. 1 point for each correct answer, 0 points for each wrong or blank answer.

1. The derivative P_x at the point labelled A is (a) positive, (b) negative, (c) zero, (d) there is not enough information to tell.
2. The derivative P_y at the point labelled A is (a) positive, (b) negative, (c) zero, (d) there is not enough information to tell.
3. The derivative Q_x at the point labelled A is (a) positive, (b) negative, (c) zero, (d) there is not enough information to tell.
4. The derivative Q_y at the point labelled A is (a) positive, (b) negative, (c) zero, (d) there is not enough information to tell.
5. The curl of $\mathbf{F}$ at the point labelled A is (a) in the direction of $+\mathbf{k}$ (b) in the direction of $-\mathbf{k}$ (c) zero (d) there is not enough information to tell.
6. The work done by the vector field on a particle travelling from point B to point C along the curve C_1 is (a) positive (b) negative (c) zero (d) there is not enough information to tell.
7. The work done by the vector field on a particle travelling from point B to point C along the curve C_2 is (a) positive (b) negative (c) zero (d) there is not enough information to tell.
8. The vector field $\mathbf{F}$ is (a) the gradient of some function f (b) the curl of some vector field $\mathbf{G}$ (c) not conservative (d) divergence free.

question	1	2	3	4	5	6	7	8
answer								


y
x
C
C1
C2
A
B

Problem 4 of 10 [12 points]

A physicist studies a vector field $\mathbf{F}$. From experiments, it is known that $\mathbf{F}$ is of the form

$$\mathbf{F} = (x-a)ye^x\,\mathbf{i} + (xe^x + z^3)\,\mathbf{j} + byz^2\,\mathbf{k},$$

where a and b are some real numbers. From theoretical considerations, it is known that $\mathbf{F}$ is conservative.

(1) [3 points] Determine a and b.

(2) [3 points] Find a potential $f(x,y,z)$ such that $\nabla f = \mathbf{F}$.

(3) [3 points] Evaluate the line intgeral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve defined by $\mathbf{r}(t) = \langle t, \cos^2 t, \cos t\rangle$, $0 \le t \le \pi$.

(4) [3 points] Evaluate the line integral

$$I = \int_C (x+1)ye^x\,dx + (xe^x + z^3)\,dy + 4yz^2\,dz,$$

where C is the same curve as in part (3). [Note: the "4" in the last term is not a misprint!].

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Problem 5 of 10 [10 points]

In the following, we use the notation $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $r = |\mathbf{r}|$, and k is some number $k = 0, 1, -1, 2, -2, \ldots$.

(1) [4 points] Find the value k for which

$$\nabla(r^k) = -3\frac{\mathbf{r}}{r^5}.$$

(2) [3 points] Find the value k for which

$$\nabla \cdot (r^k\,\mathbf{r}) = 5r^2.$$

(3) [3 points] Find the value k for which

$$\nabla^2(r^k) = \frac{2}{r^4}.$$

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Problem 6 of 10 [10 points]

Suppose the surface S is the part of the sphere $x^2 + y^2 + z^2 = 2$ that lies inside the cylinder $x^2 + y^2 = 1$ and for which $z \geq 0$.
Which of the following are parameterizations of S? Write your answer 'yes' (Y) or 'no' (N) in the following box. No explanation required. [2 points for a correct answer, 1 point if you do not answer, 0 if wrong]

	1	2	3	4	5
Y/N					

(1) $\mathbf{r}(\phi, \theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\cos\phi\,\mathbf{j} + 2\sin\phi\sin\theta\,\mathbf{k}$,
$0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \theta \leq 2\pi$.

(2) $\mathbf{r}(x, y) = x\,\mathbf{i} - y\,\mathbf{j} + \sqrt{2 - x^2 - y^2}\,\mathbf{k}$,
$x^2 + y^2 \leq 1$.

(3) $\mathbf{r}(u, \theta) = u\sin\theta\,\mathbf{i} + u\cos\theta\,\mathbf{j} + \sqrt{2 - u^2}\,\mathbf{k}$,
$0 \leq u \leq 2$, $0 \leq \theta \leq 2\pi$.

(4) $\mathbf{r}(\phi, \theta) = \sqrt{2}\sin\phi\cos\theta\,\mathbf{i} + \sqrt{2}\sin\phi\sin\theta\,\mathbf{j} + \sqrt{2}\cos\phi\,\mathbf{k}$,
$0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \theta \leq 2\pi$.

(5) $\mathbf{r}(\phi, z) = -\sqrt{2 - z^2}\sin\phi\,\mathbf{i} + \sqrt{2 - z^2}\cos\phi\,\mathbf{j} + z\,\mathbf{k}$,
$0 \leq \phi \leq 2\pi$, $1 \leq z \leq \sqrt{2}$.

Problem 7 of 10 [10 points]

Evaluate the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = (x+1)\,\mathbf{i} + (y+1)\,\mathbf{j} + 2z\,\mathbf{k}$, and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the triangle $0 \le x \le 1$, $0 \le y \le 1 - x$. S is oriented so that its unit normal has a negative z-component.

Problem 8 of 10 [10 points]

Let C be the oriented curve consisting of the 5 line segments which form the paths from $(0,0,0)$ to $(0,1,1)$, from $(0,1,1)$ to $(0,1,2)$, from $(0,1,2)$ to $(0,2,0)$, from $(0,2,0)$ to $(2,2,0)$, and from $(2,2,0)$ to $(0,0,0)$. Let

$$\mathbf{F} = (-y + e^x \sin x)\mathbf{i} + y^4\mathbf{j} + \sqrt{z}\tan z\mathbf{k}.$$

Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Problem 9 of 10 [10 points]

Let S be the part of the paraboloid $z = 2 - x^2 - y^2$ contained in the cone $z = \sqrt{x^2 + y^2}$ and oriented in the upward direction. Let

$$\mathbf{F} = (\tan\sqrt{z} + \sin(y^3))\mathbf{i} + e^{-x^2}\mathbf{j} + z\mathbf{k}.$$

Evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Problem 10 of 10 [10 points]

Which of the following statements are true (T) and which are false (F)? Write your answers in the following box. You do not need to give reasons. [1 point for a correct answer, 0.5 points if you do not answer, 0 if wrong]

	1	2	3	4	5	6	7	8	9	10
T/F										

(1) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^2)\,\mathbf{i} + \sin(t^2)\,\mathbf{j} + 2t^2\,\mathbf{k}, \quad -\infty < t < \infty,$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}, \quad -\infty < t < \infty.$$

(2) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^2)\,\mathbf{i} + \sin(t^2)\,\mathbf{j} + 2t^2\,\mathbf{k}, \quad 0 \le t \le 1,$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}, \quad 0 \le t \le 1.$$

(3) If a smooth curve is parameterized by $\mathbf{r}(s)$ where s s arc length, then its tangent vector satisfies

$$|\mathbf{r}'(s)| = 1.$$

(4) If $\mathbf{r}(t)$ defines a smooth curve C in space that has constant curvature $\kappa > 0$, then C is part of a circle with radius $1/\kappa$.

(5) If the speed of a moving object is constant, then its acceleration is orthogonal to its velocity.

(6) The vector field

$$\mathbf{F}(x,y,z) = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j} + z\mathbf{k}$$

is conservative.

(7) Suppose the vector field $\mathbf{F}(x,y,z)$ is defined on an open domain and its components have continuous partial derivatives. If $\nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is conservative.

(8) The region $D = \{\,(x,y) \mid x^2 + y^2 > 1\,\}$ is simply connected.

(9) The region $D = \{\,(x,y) \mid y - x^2 > 0\,\}$ is simply connected.

(10) If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives, then

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = 0$$

when S is the boundary of a solid region E in $\mathbb{R}^3$.