The University of British Columbia

Final Examination - April 19, 2007

Mathematics 317

Instructors: Jim Bryan and Hendryk Pfeiffer

Closed book examination   Time: 3 hours

Name ______________________ **Signature** ______________________

Student Number ________________

Special Instructions:

- Be sure that this examination has 14 pages. Write your name on top of each page.
- No calculators or notes are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Problem 1 of 10 [10 points]

Suppose the curve C is the intersection of the cylinder $x^2+y^2=1$ with the plane $x+y+z=1$.

(1.) [4 points] Find a parameterization of C.

(2.) [3 points] Determine the curvature of C.

(3.) [3 points] Find the points at which the curvature is maximum and determine the value of the curvature at these points.

Problem 2 of 10 [10 points]

Consider the curve

$$\mathbf{r}(t) = \frac{1}{3}\cos^3 t\,\mathbf{i} + \frac{1}{3}\sin^3 t\,\mathbf{j} + \sin^3 t\,\mathbf{k}.$$

(1.) Compute the arc length of the curve from $t = 0$ to $t = \frac{\pi}{2}$.

(2.) Compute the arc length of the curve from $t = 0$ to $t = \pi$.

Problem 3 of 10 [10 points]

Consider the vector field

$$\mathbf{F}(x, y, z) = -2y \cos x \sin x \, \mathbf{i} + (\cos^2 x + (1 + yz)\, e^{yz})\, \mathbf{j} + y^2\, e^{yz}\, \mathbf{k}.$$

(1.) [5 points] Find a real valued function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

(2.) [5 points] Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the arc of the curve $\mathbf{r}(t) = \langle t, e^t, t^2 - \pi^2 \rangle$, $0 \le t \le \pi$, traversed from $(0, 1, -\pi^2)$ to $(\pi, e^\pi, 0)$.

Problem 4 of 10 [10 points]

Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = \left\langle \cos z + xy^2, x\, e^{-z}, \sin y + x^2 z \right\rangle$ and S is the boundary of the solid region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Problem 5 of 10 [10 points]

Evaluate the surface integral

$$\iint_S xy^2 \, dS$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 2$ for which $x \geq \sqrt{y^2 + z^2}$.

[blank page]

Problem 6 of 10 [10 points]

Evaluate the line integral

$$\int_C (x^2 + y\,e^x)\,dx + (x\cos y + e^x)\,dy$$

where C is the arc of the curve $x = \cos y$ for $-\pi/2 \le y \le \pi/2$, traversed in the direction of increasing y.

Problem 7 of 10 [10 points]

Consider the vector field

$$\mathbf{F}(x,y,z) = \frac{x-2y}{x^2+y^2}\,\mathbf{i} + \frac{2x+y}{x^2+y^2}\,\mathbf{j} + z\,\mathbf{k}.$$

(1.) [2 points] Determine the domain of $\mathbf{F}$.

(2.) [3 points] Compute $\nabla \times \mathbf{F}$. Simplify the result.

(3.) [3 points] Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the circle of radius 2 in the plane $z = 3$, centered at $(0,0,3)$ and traversed counter-clockwise if viewed from the positive z-axis, *i.e.* viewed "from above".

(4.) [2 points] Is $\mathbf{F}$ conservative?

Problem 8 of 10 [10 points]

Suppose the curve C is the intersection of the cylinder $x^2 + y^2 = 1$ with the surface $z = xy^2$, traversed clockwise if viewed from the positive z-axis, *i.e.* viewed "from above". Evaluate the line integral

$$\int_C (z + \sin z)\, dx + (x^3 - x^2 y)\, dy + (x \cos z - y)\, dz.$$

[blank page]

Problem 9 of 10 [10 points]

A physicist studies a vector field $\mathbf{F}(x, y, z)$. From experiments, it is known that $\mathbf{F}$ is of the form

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + (axe^y z + byz)\,\mathbf{j} + (y^2 - xe^y z^2)\,\mathbf{k}$$

for some real numbers a and b. It is further known that $\mathbf{F} = \nabla \times \mathbf{G}$ for some differentiable vector field $\mathbf{G}$.

(1.) [4 points] Determine a and b.

(2.) [6 points] Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where S is the part of the ellipsoid $x^2 + y^2 + \frac{1}{4}z^2 = 1$ for which $z \geq 0$, oriented so that its normal vector has a positive z-component.

[**blank page**]

Problem 10 of 10 [10 points]

Which of the following statements are true (T) and which are false (F)? You do not need to give reasons. This problem will be graded by answer only. [1 point each]

(1.) If a smooth curve C is parameterized by $\mathbf{r}(s)$ where s is arc length, then the tangent vector $\mathbf{r}'(s)$ satisfies $|\mathbf{r}'(s)| = 1$.

(2.) If $\mathbf{r}(t)$ defines a smooth curve C in space that has constant curvature $\kappa > 0$, then C is part of a circle with radius $1/\kappa$.

(3.) Suppose $\mathbf{F}$ is a continuous vector field with open domain D. If

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every piecewise smooth closed curve C in D, then $\mathbf{F}$ is conservative.

(4.) Suppose $\mathbf{F}$ is a vector field with open domain D, and the components of $\mathbf{F}$ have continuous partial derivatives. If $\nabla \times \mathbf{F} = 0$ everywhere on D, then $\mathbf{F}$ is conservative.

(5.) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^2)\,\mathbf{i} + \sin(t^2)\,\mathbf{j} + 2t^2\,\mathbf{k}, \quad -\infty < t < \infty,$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}, \quad -\infty < t < \infty.$$

(6.) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^2)\,\mathbf{i} + \sin(t^2)\,\mathbf{j} + 2t^2\,\mathbf{k}, \quad 0 \le t \le 1,$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}, \quad 0 \le t \le 1.$$

(7.) Suppose $\mathbf{F}(x, y, z)$ is a vector field whose components have continuous second order partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

(8.) Suppose the real valued function $f(x, y, z)$ has continuous second order partial derivatives. Then $\nabla \cdot (\nabla f) = 0$.

(9.) The region $D = \{\, (x, y) \mid x^2 + y^2 > 1 \,\}$ is simply connected.

(10.) The region $D = \{\, (x, y) \mid y - x^2 > 0 \,\}$ is simply connected.