Math 316 Final Exam

April 19, 2011

Duration: 2 hours 30 minutes

Last Name: ____________________ First Name: ______________ Student Number: ________

Do not open this test until instructed to do so. Relax. This exam should have 13 pages, including this cover sheet. It is a closed book exam; no textbooks, calculators, laptops, formula sheets or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam. **Please explain your work, and circle your final solutions.** Use the extra pages if necessary.

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	**Out of**	**Score**
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
Total	100	

Problem 1 (15 points)

Consider the problem:

$$\begin{aligned}
&u_{xx} + u_{yy} - 2u = 0, \qquad 0 < x, y < \pi,\ t > 0,\\
&u(x, 0) = 0,\ u(x, \pi) = 0,\\
&u_x(0, y) = 0,\ u_x(\pi, y) = f(y).
\end{aligned}$$

Find the general form of the solution $u(x, y)$ and explain how series coefficients are to be determined in terms of $f(y)$.

Extra page (Problem 1)

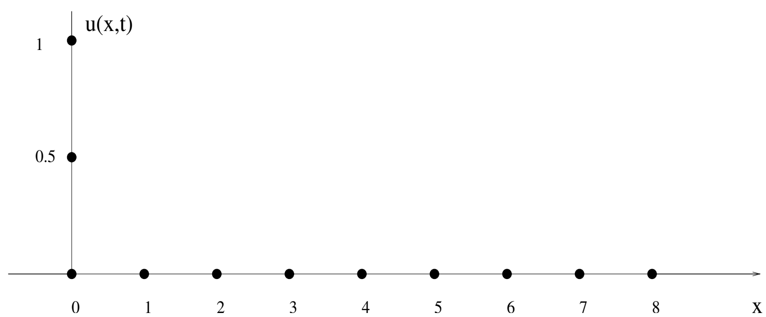
Problem 2 (15 points)

Consider the wave equation:

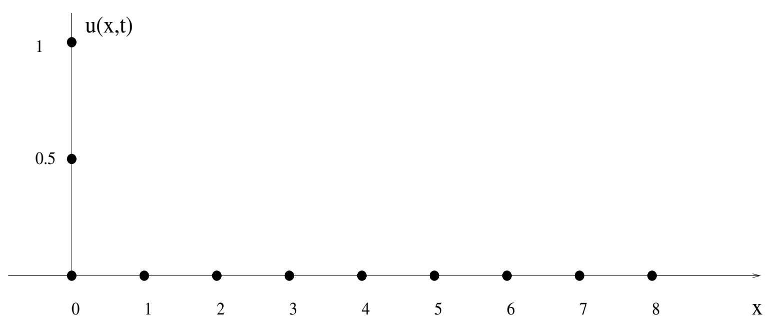
$$
\begin{aligned}
&u_{tt} = 4u_{xx}, \qquad 0 < x < 8,\ t > 0, \\
&u_x(0,t) = 0, \quad u_x(8,t) = 0, \\
&u(x,0) = f(x) = \begin{cases} 1 & 3 \le x \le 5, \\ 0 & \text{otherwise}, \end{cases} \\
&u_t(x,0) = 0.
\end{aligned}
$$

In the coordinate systems provided below, carefully sketch the solution $u(x,t)$ for $t = 0$, $t = 1$, $t = 2$, and $t = 3$.

a) $t = 0$

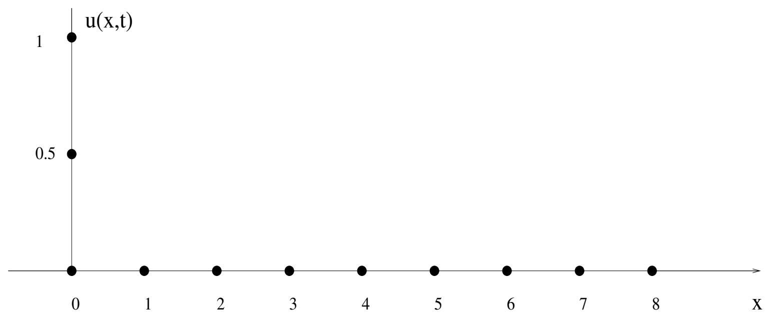


b) $t = 1$

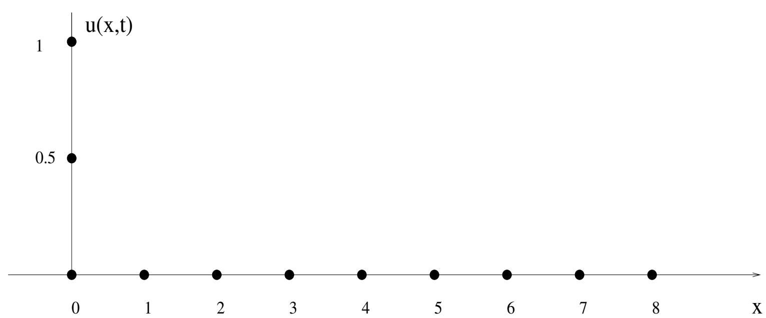


Problem 2 (continued)

c) $t = 2$



d) $t = 3$



Problem 3 (15 points)

Bacteria growth can be modeled by the equation:

$$\begin{aligned} &u_t = \alpha^2 u_{xx} + \beta^2 u, && 0 < x < 1,\ t > 0, \\ &u(0,t) = u(1,t) = 0, && \\ &u(x,0) = f(x), && \end{aligned}$$

where $u(x,t)$ is the bacteria concentration, α^2 is the diffusion rate, β^2 the growth rate, and $f(x)$ a given initial concentration. Place a condition on the values of the constants α^2 and β^2 under which $\lim_{t\to\infty} u(x,t) = 0$ *for all initial data* $f(x)$.

Extra page (Problem 3)

Problem 4 (15 points)

Consider the ordinary differential equation

$$x(1-x)y'' + (\frac{1}{2} - 2x)y' = 0, \qquad x > 0.$$

a) Show that $x = 0$ is a regular singular point. Then show that the roots of the indicial equation are $r_1 = 0$ and $r_2 = \frac{1}{2}$.

b) For $r = r_2$, find the first four non-zero terms of the power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ with $a_0 = 1$.

Extra page (Problem 4)

Problem 5 (20 points)

Consider the wave equation:

$$\begin{aligned}
&u_{tt} = 9u_{xx}, \qquad 0 < x < 1,\ t > 0,\\
&u_x(0,t) = u_x(1,t) = 0,\\
&u(x,0) = 0,\\
&u_t(x,0) = x.
\end{aligned}$$

Determine *explicitly* the solution $u(x,t)$.

Extra page (Problem 5)

Problem 6 (20 points)

Consider the following Laplace equation in the infinite domain outside the disk of radius 2:

$$
\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad 2 < r < \infty,\ -\pi < \theta < \pi,\\
&u(r,-\pi) = u(r,\pi),\\
&u_\theta(r,-\pi) = u_\theta(r,\pi),\\
&u(2,\theta) = \begin{cases} 1 & -\pi < \theta < 0, \\ 0 & 0 < \theta < \pi, \end{cases}\\
&u(r,\theta) \text{ is bounded as } r \to \infty.
\end{aligned}
$$

Determine *explicitly* the temperature $u(r,\theta)$.

Extra page (Problem 6)