

**Be sure that this examination has 10 pages, including this cover.**

**The University of British Columbia**

Final Examinations – April 2005

**Mathematics 313**

Instructor: V. Vatsal

**Time: 2.5 hours**

**Name:**

**Signature:**

**Student Number:**

**Section Number:**

Special instructions:

1. No calculators, books, notes, or other aids allowed.
2. Answer all 8 questions. All questions are worth 5 marks.
3. Give your answer in the space provided. If you need extra space, use the back of the page.
4. Show enough of your work to justify your answer. Show ALL steps.

**Problem 1:** Find all primes  $p$  for which the congruence  $X^2 + 3X + 1 \equiv 0 \pmod{p}$  has a solution.

**Problem 2:** Show that there are infinitely many primes  $p$  of the form  $p = 4k + 1$ .

**Problem 3:** Find the first 5 convergents of the continued fraction expansion for  $e = [2, 1, 2, 1, 1, 4, 1 \dots]$ .

**Problem 4:** If  $d > 1$ , show that the continued fraction expansion of  $\sqrt{d^2 - 1}$  is given by  $[d - 1, 1, 2d - 2, 1, 2d - 2 \dots]$  (the string  $1, 2d - 2$  repeats).

**Problem 5:** Let  $n$  be a positive integer. Show that there exist positive integers  $x$  and  $y$  such that  $n = x^2 - y^2$  **if and only if**  $n \not\equiv 2 \pmod{4}$ .

**Problem 6:** Find positive integers  $x$  and  $y$  such that  $x^2 + y^2 = 34255 = 5 \cdot 13 \cdot 17 \cdot 29$ .

**Problem 7:** Show that the equation  $x^2 - 5y^2 = 3z^2$  has no solutions with  $x, y, z$  nonzero integers.

**Problem 8:** Find all integer solutions to the equation  $x^2 + 2y^2 = z^2$ .