Name: UBC ID:

Math 310 - Sec. 201 - 2013 - Prof. Juan Souto

Final exam: 8:30 - 11:00

Notation. Throughout this exam, V is a complex vector spaces of finite dimension endowed with an inner product $\langle \cdot, \cdot \rangle$. The vector space of all complex polynomials is denoted by $\mathcal{P}$; the subspace consisting of those polynomials of degree at most n is denoted by $\mathcal{P}_n$.

Question 1	/38
Question 2	/25
Question 3	/20
Question 4	/15
Question 5	/17
Question 6	/20
Question 7	/15
Total	/150

Name: UBC ID:

Question 1. *Mark true or false.*

	True	False
$\{P(x) \in \mathcal{P} \vert P(-1) + P(2) = 0\}$ *is a subspace of* $\mathcal{P}$.		
$\{P(x) \in \mathcal{P} \vert P(0) = 1\}$ *is a subspace of* $\mathcal{P}$.		
$\{P(x) \in \mathcal{P} \vert \int_0^1 P(t)dt = 0\}$ *is a subspace of* $\mathcal{P}$.		
If $V \subset \mathbb{C}^n$ *is such that* $v + w \in V$ *for all* $v, w \in V$, *then* V *is a subspace.*		
Consider $\mathbb{C}^n$ *as a complex vectorspace;* $\mathbb{R}^n$ *is a subspace.*		
The union of two subspaces U_1, U_2 *of* V *is a subspace if and only if either* $U_1 \subset U_2$ *or* $U_2 \subset U_2$.		
The intersection of three subspaces of V *is a subspace.*		
A vector space has infinite dimension if and only if it contains a subspace of dimension n *for all* n.		
If $W \subset V$ *is a subspace with* $\dim(W) = \dim(V)$ *then* $W = V$.		
If $W_1, W_2 \subset V$ *are subspaces with* $\dim(W_1) = \dim(W_2)$ *then* $W_1 = W_2$.		
$T : \mathcal{P} \to \mathcal{P}, \quad T(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1x + a_2x^2$ *is linear.*		
$T : \mathcal{P} \to \mathbb{C}^3, \quad T(P(x)) = P(1) + P(2) - P(3)$ *is linear.*		
$T : \mathcal{P} \to \mathbb{C}, \quad T(a_0 + a_1x + \cdots + a_nx^n) = a_0^2$ *is linear.*		
$T : \mathcal{P} \to \mathbb{C}, \quad T(a_0 + a_1x + \cdots + a_nx^n) = a_0 - \bar{a}_1 + a_2 - \bar{a}_3$ *is linear.*		
If $T : V \to V$ *is linear and maps a basis of* V *to a basis of* V, *then* T *is invertible.*		
The linear map $T : \mathcal{P}_2 \to \mathbb{C}^3$, $T(P(x)) = (P(1), P(2), P(10))$ *is invertible.*		
If $W_1, W_2 \subset V$ *are subspaces with* $\dim(W_1) = \dim(W_2)$ *then there is a linear map* $T : V \to V$ *with* $T(W_1) = W_2$.		

	True	False
Every linear map $T : V \to V$ has an eigenvalue.		
$T : V \to V$ *is surjective if and only if* $\mathrm{Ker}(T) = 0$.		
If the image of a linear map $T : \mathcal{P}_2 \to \mathcal{P}_4$ contains 3 linearly independent polynomials, then T is injective.		
There is a surjective linear map $T : \mathcal{P}_2 \to \mathcal{P}_4$		
For every $d = 0, 1, \ldots, 5$ there is a linear map $T : \mathcal{P}_4 \to \mathcal{P}_4$ whose kernel has dimension d.		
If the kernel of a linear map $T : \mathcal{P}_n \to \mathcal{P}_{n-2}$ has dimension 7 then T is surjective.		
There is an injective linear map $T : \mathcal{P}_n \to \mathcal{P}_{n-2}$.		
There is a unique matrix associated to every linear map $T : V \to W$.		
$T : V \to V$ is diagonalizable if and only if all eigenvalues of T are distinct.		
If all eigenvalues of T are distinct, then T is diagonalizable.		
There is a basis with respect to which the matrix of T is upper triangular.		
T is injective if and only if 0 *is not an eigenvalue of T.*		
T has an eigenvalue if and only if T is normal.		
If T is normal, then T is diagonalizable.		
If T is normal, then there is a ON-basis of V consisting of eigenvectors.		
$\lambda \in \mathbb{C}$ is an eigenvalue of T if and only if $\mathrm{Ker}((T - \lambda\,\mathrm{Id})^5) \neq 0$.		
If T is normal and v is an eigenvector of T, then v is also an eigenvector of T^*.		
If $T^5 = 0$, then $T = 0$.		
If T^5 is diagonalizable, then T is diagonalizable.		
Let T^* be the adjoint of T. If $T^* = 0$, then $T = 0$.		
The matrix of T^* with respect to an arbitrary basis of V is the transpose conjugate of that of T.		

Name: UBC ID:

Question 2. *Let $T : V \to V$ be a linear map.*

(1) *Define the* kernel[1] $\mathrm{Ker}(T)$ *of* T.

(2) *Prove that* $\mathrm{Ker}(T)$ *is a subspace of* V.

(3) *Prove that* T *is injective if and only if* $\mathrm{Ker}(T) = 0$.

[1]or equivalently, the *nullspace*.

(4) *Give an example of a linear map* $T : V \to V$ *with* $\mathrm{Ker}(T) \neq \mathrm{Ker}(T^2) \neq \mathrm{Ker}(T^3)$.

(5) *Suppose that* $\mathrm{Ker}(T) = \mathrm{Ker}(T^2)$. *Prove that* $\mathrm{Ker}(T^2) = \mathrm{Ker}(T^3)$.

Name: UBC ID:

Question 3. *Given* $x_0, \ldots, x_n, y_0, \ldots, y_n \in \mathbb{C}$ *suppose that* $x_i \neq x_j$ *for* $i \neq j$*. Prove that there is a unique polynomial* $P(x) \in \mathcal{P}_n$ *of degree at most* n *satisfying* $P(x_i) = y_i$ *for all* $i = 0, \ldots, n$*.*

Name: UBC ID:

Question 4.

(1) *Let* $v_1, \dots, v_r \in V$. *Define* $(v_1, \dots, v_r)$ is linearly independent.

(2) *Let* $T : V \to V$ *be linear. Suppose that* $v_1 \in \mathrm{Ker}(T^2)$, $v_2 \in \mathrm{Ker}(T - \mathrm{Id})$ *and* $v_3 \in \mathrm{Ker}(T + \mathrm{Id})$ *are non-zero vectors. Prove that* (v_1, v_2, v_3) *is linearly independent.*

Name: UBC ID:

Question 5. *Let $T : V \to V$ be linear and $(v_1, \ldots, v_d)$ a basis of V. Prove that the following statements are equivalent:*

(1) *The matrix of T with respect to the basis $(v_1, \ldots, v_d)$ is upper triangular.*
(2) $T(v_j) \in \mathrm{Span}(v_1, \ldots, v_j)$ *for all $j = 1, \ldots, d$.*

Name: UBC ID:

Question 6. *Let $T : V \to V$ be linear.*

(1) *Suppose that $T : V \to V$ is diagonalizable. Prove that there is $S : V \to V$ linear with $S^{\dim(V)} = T$.*

(2) *Give an example of a complex vector space V of finite dimention and of a non-zero operator $T : V \to V$ with $T^{\dim(V)} = 0$.*

(3) *Suppose that* $T : V \to V$ *is a non-cero operator with* $T^{\dim(V)} = 0$. *Prove that there is no operator* $S : V \to V$ *with* $S^{\dim(V)} = T$.

Name: UBC ID:

Question 7. *Let* $T : V \to V$ *be linear.*

(1) *Define* T is normal.

Suppose from now on that $T : V \to V$ *is normal and recall that this implies that* $\|T(v)\| = \|T^*(v)\|$ *for all* $v \in V$*.*

(2) *Prove that* $v \in V$ *is an eigenvector of* T *if and only if it is an eigenvector of* T^**.*

(3) *Suppose that* $v \in V$ *is an eigenvector of* T. *Prove that the orthogonal complement of* $\mathrm{Span}(v)$ *is* T*-invariant.*

Name: *UBC ID:*

Name: *UBC ID:*