

The University of British Columbia

Final Examination - December 02, 2014

Mathematics 308

Closed book examination. No calculators.

Time: 2.5 hours

Last Name _____ First _____ Signature _____

Student Number _____

No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action.
 - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
 - (b) Speaking or communicating with other candidates.
 - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

| Problem | Points | Score |
|---------|--------|-------|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 20 | |
| Total: | 100 | |

1. Let $A = (3, 2, -2)$, $B = (4, 4, -3)$, $C = (6, 1, -1)$ be three points on a plane P in \mathbb{R}^3 .

(a) (5 points) Find the equation form of the plane P .

(b) (2 points) Do the points A, B, C and $D = (4, -3, 1)$ lie on the same plane? Justify your answer.

(c) (3 points) Find the area of $\triangle ABC$.

2. Let L_1, L_2 be parallel lines in \mathbb{R}^3 . Suppose that $A = (4, -3, 7)$ is a point on L_1 and an equation form of L_2 is given by
$$\begin{cases} x - 7y + 2z = 0 \\ x - 4y + z = 0. \end{cases}$$

(a) (4 points) Find the parametric form of L_1 .

(b) (6 points) Find the distance between L_1 and L_2 .

3. (a) (3 points) Suppose the line $L(t) = (1, 3, a) + t(1, 2, b)$ lies entirely on the plane $P : x + 2y - z = 10$. Find the values of the constants a and b .

- (b) (4 points) Suppose that $A = (1, 1, 2)$ is a vertex of a parallelepiped in \mathbb{R}^3 and $B = (2, 1, 3)$, $C = (3, 2, 2)$ and $D = (0, 2, 6)$ are the three adjacent vertices of A . Find the coordinates of the vertex E opposite to A .

- (c) (3 points) Let θ be the acute angle between the line $(-5, 3, 2) + t(1, 1, 1)$ and the plane $2x + 2y - z = 7$. Find $\cos \theta$.

4. (a) (3 points) State Playfair's axiom in plane geometry.

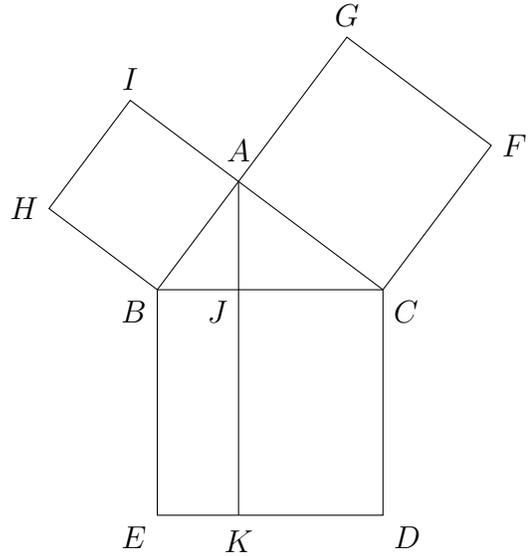
(b) (2 points) Determine whether each of the following statements is true or false in a geometry that satisfies all the axioms, except for postulate 5, in Euclid's *Elements*.

True/False

- i. We can construct an equilateral triangle. _____
- ii. Given a straight line L and a point A not on it, there exists a straight line parallel to L through A . _____
- iii. If a straight line falling on two straight lines L_1 and L_2 makes the alternate angles equal to one another, then L_1, L_2 are parallel. _____
- iv. Straight lines parallel to the same straight line are also parallel. _____

(c) (5 points) Assuming all the axioms in *Elements*, prove that the sum of interior angles of a triangle is two right angles. You can use all the definitions, postulates, common notions and propositions listed on the last page of the exam paper. Justify each step in your argument carefully and state all the axioms and propositions used.

5. (10 points) In the figure below, $\triangle ABC$ has a right angle at A and $BCDE$, $ACFG$, $ABHI$ are squares on the three sides $\triangle ABC$. AJK is a straight line parallel to CD . Prove that the quadrilaterals $ACFG$ and $CDKJ$ have equal area. You can use all the definitions, postulates, common notions and propositions listed on the last page of the exam paper. Justify each step in your argument carefully and state all the axioms and propositions used. (Hint: Join AD and BF)



6. (a) (6 points) Find an unit vector on the axis of rotation of the rotary reflection T defined by $T(\vec{x}) = M\vec{x}$, where

$$M = \frac{1}{9} \begin{pmatrix} -8 & -1 & -4 \\ -1 & -8 & 4 \\ 4 & -4 & -7 \end{pmatrix}$$

- (b) (4 points) Show that the composition $S \circ T$ of two isometries $S, T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is also an isometry.

7. (a) (5 points) Determine whether the isometry defined by each orthogonal matrix below is a reflection across a line or a plane, a rotation or a rotary reflection. Justify your answer. You do not need to find the plane/line/angle of reflection/rotation.

i. $\frac{1}{5} \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}$

ii. $\frac{1}{13} \begin{pmatrix} -12 & 0 & 5 \\ 0 & 13 & 0 \\ -5 & 0 & -12 \end{pmatrix}$

- (b) (5 points) Find the matrix associated to the reflection $\text{Ref}_L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where L is the line $2x - 5y = 0$ in \mathbb{R}^2 .

8. For the following short questions, you only need to write down the correct answers for full credit. However, partial credit will be awarded for incorrect answer with substantially correct explanations.

(a) (3 points) List all the possible numbers of faces of Platonic solids? How many edges and vertices does the Platonic solid with the most number of faces have?

(b) (3 points) List all the n among $60 \leq n \leq 120$ such that regular n -gons are constructible by a compass and a straightedge.

(c) (4 points) i. Which of the following complex numbers are constructible?

| | | | | |
|---------|---|--------------|----------------|----------------|
| z | 7 | $-\sqrt{2}i$ | $e^{5\pi i/7}$ | $\cos(3\pi/5)$ |
| Yes/No? | | | | |

ii. Given a figure of a regular 49-gon, a compass and a straightedge, which of the following regular n -gons can be constructed?

| | | | | | | |
|---------|---|----|----|----|----|-----|
| n | 7 | 14 | 21 | 63 | 85 | 343 |
| Yes/No? | | | | | | |

9. Do the following constructions using compass and straightedge. Show all the lines and circles you draw in the intermediate steps. You can give brief explanations for your constructions but they are not required.

(a) (5 points) Trisect the line AB below. Label the trisection points by X, Y .

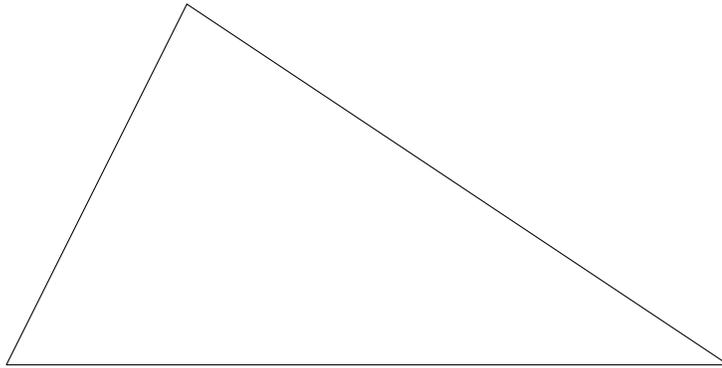


(b) (5 points) You are given a line AB of unit length below.



Use this scale to construct a line of length $\sqrt{10}$ and label the endpoints by X, Y .

- (c) (5 points) Construct the inscribed circle of the triangle. Label its center by I .



- (d) (5 points) Given a point A and a line BC below, construct a line AX with length equal to BC using Euclid's collapsing compass and a straightedge. In other words, you can only use a line from the center as the radius when constructing circles (or arcs) as in postulate 3 of *Elements*. You do not need to state the axioms you use.

A .

\overline{BC}

A partial list of axioms and propositions from Book I of Euclid's *Elements*

Definition 15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definition 21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Definition 22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral. . .

Definition 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and radius.

Postulate 4. That all right angles equal one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common notion 1. Things which equal the same thing also equal one another.

Common notion 2. If equals are added to equals, then the wholes are equal.

Common notion 3. If equals are subtracted from equals, then the remainders are equal.

Common notion 4. Things which coincide with one another equal one another.

Common notion 5. The whole is greater than the part.

Proposition 4. If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Proposition 13. If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Proposition 14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Proposition 15. If two straight lines cut one another, then they make the vertical angles equal to one another.

Proposition 23. To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Proposition 26. If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

Proposition 29. A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Proposition 31. To draw a straight line through a given point parallel to a given straight line.

Proposition 38. Triangles which are on equal bases and in the same parallels equal one another.

Proposition 41. If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.