The University of British Columbia

Final Examination - April 24, 2010

Mathematics 307

Section 201

Closed book examination   Time: 2.5 hours

Last Name ⎯⎯⎯⎯⎯⎯⎯⎯   **First Name** ⎯⎯⎯⎯⎯⎯⎯⎯

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Rules governing examinations

question	score	out of
1		20
2		20
3		20
4		20
5		20
6		20
total		100

Special instructions

1. Solve 5 out of 6 questions to have a full score. All questions are equal in their weight.

2. Indicate which 5 questions are intended for grading by marking the question numbers in the table above. The draft of the remaining question, if any, will not be graded.

3. Make sure to explain any notation introduced in your solution and not given in the formulation of the question.

4. Provide complete derivation, as the final answers only are not worth any score.

5. No auxiliary material of any kind, including calculators, is allowed.

6. Submit the question sheet along with your answers. Make sure that all of your written pages remain intact.

7. Verify that your full name, student number and signature appear on the work in ink.

1. Below are 6 statements. Mark each one as true or false and give a general explanation ($3\frac{1}{3}$ points per part). If you decide that a statement is false, you can give a contradicting example instead. Bear in mind that to mark a statement true all of its parts must be true.

   (a) It is given that a real matrix A is a root of the equation $A^n - A = 0$ with $n > 1$ an integer. Then it is possible to conclude the following. The size of A is $n \times n$ because by the Cayley - Hamilton theorem A satisfies $p(A) = 0$, with $p(\lambda)$ being the characteristic polynomial of A. Since it is given that A is a root of an n-th degree polynomial, its characteristic polynomial must be of degree n and thus A is $n \times n$.

   (b) For any two distinct vectors $r_1, r_2 \in \mathbb{R}^4$ there exists a subspace $V \subset \mathbb{R}^4$ such that $V \perp \text{span}\{r_1, r_2\}$. In addition there exists a set of vectors $W \subset \mathbb{R}^4$, such that each vector $w \in W$ satisfies $w \notin V$ and $\langle w, r_i \rangle = 0$, $i = \{1, 2\}$. Then the set W is
   $$W = \mathbb{R}^4 \setminus \{V \cup \text{span}\{r_1, r_2\}\}$$
   and its dimension is
   $$\dim W = 1 \quad \text{or} \quad \dim W = 2.$$

   (c) If the equation $Ax = 0$ with $A \in \mathbb{R}^{m \times n}$ has a non-trivial solution, then $\dim \text{Im } A^T \leqslant n - 1$.

   (d) The characteristic polynomial of a matrix A is given by $p(\lambda) = \lambda^2 - \lambda$. Then using the Cayley - Hamilton theorem it is possible to compute $A^{-1} = I$.

   (e) A projection on a subspace spanned by the columns of a real matrix A is given by $P = A(A^T A)^{-1} A^T$. Then if A is an orthogonal matrix, $P = AA^T$.

   (f) $P_{n \times n}$ is a projection matrix with $n \geqslant 2$. Then regardless of n it is always possible to construct a quadratic polynomial such that P is its root.

2. A real matrix $A_{5 \times 5}$ is brought into its Jordan canonical form
$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

   by a matrix $S_{5 \times 5}$, i.e. $A = SBS^{-1}$. Answer the questions below, substantiate all your statements ($3\frac{1}{3}$ points per part).

   (a) What are the eigenvalues of A? Include algebraic multiplicities.

   (b) What is the rank of $A - I$ (I being the 5×5 identity matrix)?

   (c) What is the dimension of the kernel of $A + I$ (I being the 5×5 identity matrix)?

   (d) What is the determinant of the product matrix CA if $C \in \mathbb{C}^{5 \times 5}$ is a Hermitian matrix with full geometric multiplicity?

   (e) What is the condition of A?

   (f) How should the matrix S be computed? Write an equation for each column, state whether a unique solution is expected and explain.

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3. It is requested to construct an approximating series for the function below. The goal is to sum the series on a computer and thus it is required to

- have the highest achievable precision at all possible points x
- involve computations with real numbers only.

$$f(x) = \begin{cases} 1 & 0 < x < \dfrac{\pi}{2} \\ \pi & \dfrac{\pi}{2} < x < \pi \\ \alpha & x = 0, \dfrac{\pi}{2}, \pi, \end{cases}$$

with α being a real parameter to be input by the user along with the number of terms N to be summed. Use the guidelines below to complete the task (5 points per part).

(a) Suggest an appropriate basis based on the constraints above. You are allowed to extend the function as necessary to enhance the computing performance. Explain your choice.

(b) Write the function as an exact series with the proposed basis and compute the coefficients. You may use the following trigonometric formula as given: $\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

(c) Structure a code layout to sum N non-vanishing terms of the series. Explain the choice of all arbitrary parameters in your code. You may assume N even or odd for convenience, but explain your choice.

(d) What values does the series yield at $x = 0, \frac{\pi}{2}, \pi$? How could the expansion be modified to give the value of α exactly? Explain. You may leave your answer as a series if it is not summable directly.

4. Consider the following recursion relation

$$x_{n+1} = ax_n + bx_{n-1} \qquad n \geqslant 1, \quad a, b \in \mathbb{R}.$$

(a) (5 points) Find the matrix A_n so that for all n

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A_n \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}.$$

(b) (5 points) How must the parameters a and b be related if it is given that A_1 has $\lambda_1 = 1$ as its eigenvalue? What is the other eigenvalue of A_1 (in terms of a and b)?

(c) (10 points) Assume that $\lambda_1 \neq \lambda_2$. What subspace must the initial vector $\begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$ belong to for the sequence $\{x_0, x_1, x_2, \ldots\}$ to decay, i.e. have

$$\lim_{n \to \infty} x_n = 0?$$

Set the appropriate condition on b or a, and identify the subspace.

5. Consider a matrix $A \in \mathbb{C}^{n \times n}$, a real diagonal matrix S^2 containing the eigenvalues of A^*A (all positive) and a matrix V that diagonalises A^*A. Answer the following questions, substantiate all your statements (4 points per part).

   (a) Explain why S^{-1} exists and show that the matrix $U = AVS^{-1}$ is unitary.

   (b) Show that U diagonalises AA^* with S^2 giving the eigenvalues.

   (c) From (a) and (b) it follows that AA^* and A^*A have the same eigenvalues. How are their eigenvectors connected?

   (d) For the example below

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\ -\dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & 0 & \dfrac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

   what is $||A||$? Explain. Note that here $A \notin \mathbb{C}^{n \times n}$.

   (e) Find a vector on the unit circle $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ that is stretched most by A.

6. Consider the matrix

$$P = \begin{pmatrix} \dfrac{1}{2} & 0 & 0 & 0 \\ \dfrac{1}{6} & \alpha & 0 & * \\ \dfrac{1}{6} & 0 & \beta & 0 \\ \dfrac{1}{6} & * & * & \gamma \end{pmatrix}.$$

   (a) (5 points) Fill in the star etnries so that P is a stochastic matrix.

   (b) (5 points) Show that the eigenvalues of P are

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \beta, \quad \lambda_3 = \alpha + \gamma - 1, \quad \lambda_4 = 1.$$

   (c) (10 points) Given that $\beta = 1$ and $\alpha = \gamma = \dfrac{3}{4}$, draw the network corresponding to the random walk represented by P and find the limiting state x_n of the walk. Explain the result according to the drawing.