Final Exam
Friday, December 15, 2006.

No books, notes or calculators

Part I. These problems are common to Sections 101 and 102

Problem 1. (10 points)
(The scalar field is $\mathbb{R}$.) Consider the matrix A and column vector $\vec{b}$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(a) (6 points) Find the $PA = LU$ factorization for A.

(b) (2 points) Recall that the $PA = LU$ factorization can be used to rewrite the system $A\vec{x} = \vec{b}$ as two systems with triangular coefficient matrix. Write down these two triangular systems.

(c) (2 points) Solve the two triangular systems to find all $\vec{x}$ such that $A\vec{x} = \vec{b}$.

Problem 2. (10 points)
(The scalar field is $\mathbb{R}$.) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 2 & 4 & 3 & 0 & 5 \\ 1 & 2 & -1 & 5 & 5 \end{pmatrix}$$

(a) Find a basis for the nullspace of A.

(b) Find a basis for the column space of A.

(c) Find a basis for the left nullspace of A.

(d) Find a basis for the row space of A.

Problem 3. (10 points)
Consider the discrete dynamical system

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$$

which describes the evolution of the vector $\begin{pmatrix} a & b & c \end{pmatrix}^T$.

(a) (2 points) Explain what a Markov matrix (stochastic matrix) is and why the coefficient matrix of this discrete dynamical system is a Markov matrix.

(b) (6 points) Find a fixed vector for this discrete dynamical system, i.e., a vector $\begin{pmatrix} a & b & c \end{pmatrix}^T$ such that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix},$$

for all n.

(c) (2 points) Find the limit of the vector $\begin{pmatrix} a_n & b_n & c_n \end{pmatrix}^T$ as $n \to \infty$ when the initial vector is

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 39 \end{pmatrix}.$$

Problem 4. (15 points)
Consider the real matrix

$$A = \begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix}$$

(a) (7 points) use complex numbers to diagonalize A, this means to find matrices S, Λ, and S^{-1}, such that

$$A = S\Lambda S^{-1},$$

where Λ is diagonal.

(b) (2 points) Write down (using complex numbers) the general solution of the system of differential equations

$$\frac{d}{dt}\vec{x}(t) = A\,\vec{x}(t).$$

(c) (2 points) Discuss the long term behaviour of the solutions of $\vec{x}' = A\,\vec{x}$.

(d) (2 points) Solve the initial value problem

$$x_1'(t) = 2x_1(t) + x_2(t) \quad x_2'(t) = -8x_1(t) - 2x_2(t)\,; \qquad x_1(0) = 3 \quad x_2(0) = -2\,.$$

(e) (2 points) Write down a (real!) basis for the real vector space of all differentiable functions $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^2$, satisfying $\vec{x}'(t) = A\,x(t)$, for all t.

**Problem 5.** (8 points)
Consider the real matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) (2 points) Explain what property the matrix A has, which assures that you can diagonalize it without the help of complex numbers.

(b) (6 points) find a real matrix S and a diagonal matrix D, such that

$$A = SDS^T.$$

Part II. These problems are specific to Section 101.

Problem 6. (12 points)
Consider the following graph:
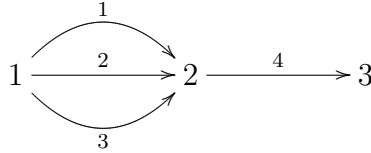
(a) (2 points) Write down the edge-node incidence matrix A for this graph. (Do not ground any node yet!)

(b) (2 points) Write down loop vectors which give a basis of the left nullspace of A. (These should be given as row vectors.)

(c) (2 points) Sketch a spannning tree of the graph and write down the corresponding basis for the row space of A.

(d) (6 points) Ground Node Number 1, put a battery of value 2 (Volt) on Edge Number 2 ($b_2 = 7$), add an external current (outflow or overflow) of value 4 (Ampere) at Node Number 3 ($f_3 = 4$), and assume every edge has unit resistance (1 Ohm). Determine the induced potentials at the 3 nodes and the currents on the 4 edges. Use the formulas:

$$C(-A\vec{x} + \vec{b}) = \vec{y} \qquad A^T \vec{y} = \vec{f},$$

where $\vec{x}$ is the potential vector, $\vec{y}$ is the current vector, and C is the conductance matrix.

Problem 7. (12 points)
Assume a linear code C in $\mathbb{F}_2^5$ is given by the generating matrix

$$G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) What is the dimension of the code C?

(b) List all the code words in C.

(c) Find the minimal Hamming distance between code words.

(d) How many errors can C detect? correct?

(e) Write down a parity check (or Hamming matrix) H for the code C.

(f) Suppose you receive the words $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$ and $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \end{pmatrix}^T$ through a noisy channel. Decode these words.

(g) what assumption do you have to make so that you can be sure that your answers to (f) are indeed the code words which were sent through the noisy channel?

Problem 8. (8 points)

Find the singular value decomposition $A = U\Sigma V^T$ for the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Part II. These problems are specific to Section 102.

Problem 6. (12 points)
Consider the real matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) (6 points) Find the QR factorization of A.

(b) (3 points) Find the least-squares 'solution' to $A\vec{x} = \vec{b}$, where

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}.$$

(c) (3 points) Find the projection of $\vec{b}$ onto the column space of A.

Problem 7. (12 points)
Decide whether or not the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) The product of two orthogonal matrices is orthogonal.

(b) If $AB = I$, then $BA = I$. (Warning: do *not* assume that A and B are square matrices.)

(c) For every 8×5 matrix A,
$$\dim N(A) \geq 3.$$

(d) For every 5×5 matrix A with real entries,
$$\det(A^T A) \geq 0.$$

(e) If $A\vec{x} = \vec{b}$ and $A^T \vec{y} = \vec{0}$, then $\vec{y}^T \vec{b} = 0$.

Problem 8. (8 points)
Let

$$A = \begin{pmatrix} 0 & 1 & 1 & x \\ 1 & 0 & x & 1 \\ 1 & x & 0 & 1 \\ x & 1 & 1 & 0 \end{pmatrix}$$

(a) Calculate the determinant $\det A$.

(b) Find all x such that $\det A = 0$.