The University of British Columbia

Final Examination - April 2014

Mathematics 303
Dr. D. Brydges

Time: 2.5 hours

Last Name ____________________ **First** ______________ **Signature** ____________________

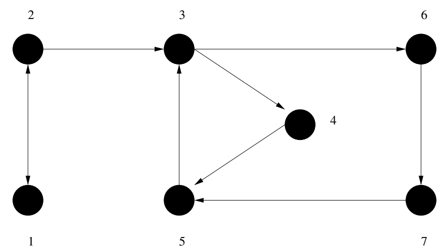
Student Number ______________________

Special Instructions:

Closed book exam, no calculators. Brief explanation is required whenever it is not clear how answers are obtained. The test has 7 questions and is out of 70. **Be sure this exam has 15 pages including the cover**.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

1. Consider the Markov chain $(X_n,\, n = 0, 1, \dots)$ whose states and transitions of positive probability are indicated by the oriented arrows in the diagram. One arrow has both orientations; the others only permit transitions in one direction.



(a) (2 points) What are the communicating classes?

Solution: $\{1, 2\}$, $\{3, 4, 5, 6, 7\}$.

(b) (1 point) Which states, if any, have period 2?

Solution: $\{1, 2\}$

(c) (1 point) Which states, if any, have period 3?

Solution: None.

(d) (1 point) Which states, if any, are transient?

Solution: $\{1, 2\}$.

(e) (3 points) What are the hypotheses in the main theorem about when the long term probability $\lim_{n\to\infty} \mathbb{P}_i(X_n = j)$ exists and does not depend on i.

Solution: The Markov chain $(X_n,\, n = 0, 1, \ldots)$ is irreducible, aperiodic and positively recurrent.

(f) (1 point) Do the hypotheses you stated in part (e) hold for $(X_n,\, n = 0, 1, \ldots)$?

Solution: No. Not irreducible.

(g) (1 point) Explain why $\lim_{n\to\infty} \mathbb{P}_i(X_n = j)$ exists and discuss whether it depends on the initial state i.

Solution: It exists because we can apply the main theorem to the Markov chain restricted to the communicating class $\{3, 4, 5, 6, 7\}$. When $i = 1, 2$ the limit is zero because these states are transient.

2. Let $(X_n, n = 0, 1, \ldots)$ be a Markov chain with transition matrix P_{ij}.

   (a) (2 points) Let $\pi = (\pi_i)$ be a stationary distribution. What equation expresses the fact that π is stationary?

   Solution: $\sum_i \pi_i P_{ij} = \pi_j$ for all states j.

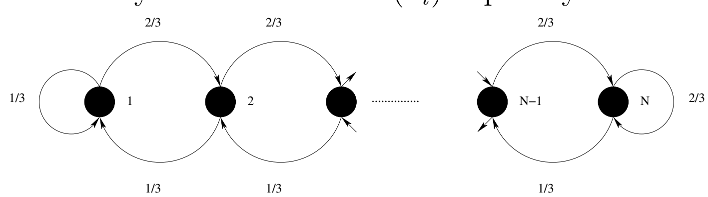
   (b) (2 points) Let $x = (x_i)$ be a probability distribution that satisfies the equations of detailed balance. What are these equations?

   Solution: $x_i P_{ij} = x_j P_{ji}$ for all states i, j.

(c) (1 point) Prove that a probability distribution $x = (x_i)$ that solves the equations of detailed balance is stationary.

Solution: $\sum_i x_i P_{ij} = \sum_i x_j P_{ji} = x_j$.

(d) (5 points) For the Markov chain with states $1, 2, \ldots, N$ with transition diagram as shown, find the stationary distribution $\pi = (\pi_i)$ explicitly.



Solution: The equations of detailed balance in the case of this problem are $\pi_i \frac{2}{3} = \pi_{i+1} \frac{1}{3}$ for $i = 1, \ldots, N-1$. This simplifies to $\pi_{i+1} = 2\pi_i$. Therefore $\pi_i = 2^{i-1}\pi_1$. π_1 must be chosen so that $\pi_1 + 2\pi_1 + \cdots + 2^{N-1}\pi_1 = 1$. This is the same as $\pi_1 \frac{1-2^N}{1-2}$. Therefore $\pi_1 = \frac{1}{2^N - 1}$ and $\pi_i = \frac{2^{i-1}}{2^N-1}$.
It is possible but harder to do this problem by solving $\pi P = \pi$ by looking for a solution of the form $\pi_i = \lambda^i$ as in gamblers ruin.

3. Let S_n be the position at time n of symmetric simple random walk on $\mathbb{Z}$ with $S_0 = 0$. Recall that $S_n = X_1 + \cdots + X_n$ where $X_1, \ldots, X_n$ are independent steps which have probability $\frac{1}{2}$ to be 1 and -1. Let $N^{(99)}$ be the number of visits the random walk makes to the origin up to and including time 99, that is $N^{(99)} = \#\{n = 0, ..., 99 | S_n = 0\}$.

   (a) (1 point) Write $N^{(99)}$ in terms of the indicator functions $I_{S_0=0}, I_{S_1=0}, \ldots, I_{S_{99}=0}$.

   Solution: $N^{(99)} = \sum_{n=0}^{99} I_{S_n=0}$.

   (b) (1 point) Express $\mathbb{E}N^{(99)}$ in terms of $\mathbb{P}(S_n = 0)$ and show why your answer is correct.

   Solution:

$$\mathbb{E}N^{(99)} \overset{(a)}{=} \mathbb{E}\sum_{n=0}^{99} I_{S_n=0} \tag{1}$$

$$= \sum_{n=0}^{99} \mathbb{E}I_{S_n=0} = \sum_{n=0}^{99} \mathbb{P}(S_n = 0). \tag{2}$$

   (c) (2 points) Find an explicit formula for $\mathbb{E}e^{ikX_1}$.

   Solution: $\mathbb{E}e^{ikX_1} = e^{ik}\mathbb{P}(X_1 = 1) + e^{-ik}\mathbb{P}(X_1 = -1) = \frac{1}{2}\Big(e^{ik} + e^{-ik}\Big)$.

(d) (2 points) Let $r(k) = \mathbb{E}e^{ikX_1}$. Express $\mathbb{E}e^{ikS_n}$ in terms of $r(k)$.

Solution:

$$\mathbb{E}e^{ikS_n} = \mathbb{E}\Big(e^{ikX_1}\dots e^{ikX_n}\Big) = \mathbb{E}\Big(e^{ikX_1}\Big)\dots\mathbb{E}\Big(e^{ikX_n}\Big) = \big(r(k)\big)^n.$$

(e) (2 points) What is $\int_{-\pi}^{\pi} e^{ikS_n}\,dk$ when $S_n = 0$ and what is it when $S_n = s$ where s is a nonzero integer? Hint. $e^{iks} = \cos ks + i \sin ks$.

Solution: For $k = 0$, $\int_{-\pi}^{\pi} e^{ikS_n}\,dk = \int_{-\pi}^{\pi} 1\,dk = 2\pi$. For $S_n = s$ it is zero: $\int_{-\pi}^{\pi} e^{iks}\,dk = \frac{1}{s}\Big[\sin(ks) - i\cos(ks)\Big]_{k=-\pi}^{k=\pi} = 0$ because $\sin s\pi = \sin s(-\pi)$ and likewise cos.

(f) (2 points) Show that $\mathbb{E}N^{(99)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^{100}}{1-r}\,dk$ where $r = r(k)$ and $r(k)$ was defined in part (d).

Solution:

$$\begin{aligned}
\mathbb{E}N^{(99)} &\overset{(b)}{=} \sum_{n=0}^{99} \mathbb{E}I_{S_n=0} \overset{(e)}{=} \sum_{n=0}^{99} \mathbb{E}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikS_n}\,dk\Big) \\
&= \sum_{n=0}^{99} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{E}\Big(e^{ikS_n}\Big)\,dk \overset{(d)}{=} \sum_{n=0}^{99} \frac{1}{2\pi}\int_{-\pi}^{\pi} r^n\,dk \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=0}^{99} r^n\,dk = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^{100}}{1-r}\,dk
\end{aligned}$$

4. A computer virus spreads by forcing an infected computer to transmit copies of the virus to other computers. Let X_n be the number of infected computers at time n. Assume that $X_0 = 1$ and that $(X_n, n = 0, 1, \dots)$ is a branching process where each infected computer independently infects a random number Z of other computers. The distribution for Z is given by $\mathbb{P}(Z = n) = p^n(1-p)$ where $n = 0, 1, \dots,$ and p is a parameter between 0 and 1. Notice that $\sum_{n=0}^{\infty}(1-p)p^n = (1-p)\sum_{n=0}^{\infty} p^n = (1-p)\frac{1}{1-p} = 1$.

   (a) (4 points) Give the definition of the generating function $G_1(s)$ of X_1, which is an infinite sum in powers of s, and write out the first three terms in this sum.

   Solution: $G_1(s) = \sum_{x=0}^{\infty} s^x \mathbb{P}(X_1 = x) = (1-p) + (1-p)ps + (1-p)(ps)^2 + \dots$

   (b) (2 points) Simplify your answer in part (a). The sum can be evaluated.

   Solution: $G_1(s) = (1-p)\Big(1 + ps + (ps)^2 + \dots\Big) = (1-p)\frac{1}{1-ps}$ for $ps < 1$.

(c) (4 points) Find the probability as a function of p, that the virus goes extinct. If you could not do earlier parts explain how to find the extinction probability.

Solution: Must find smallest nonnegative solution η to $G_1(s) = s$. From part (b), $(1-p)\frac{1}{1-ps} = s$, $ps^2 - s + 1 - p = 0$, $s = 1, \frac{1-p}{p}$ so $\eta = \min(1, \frac{1-p}{p})$.

5. Let Q_{ij} be the transition matrix for simple random walk on $\mathbb{Z}$ which steps to the right with probability $\frac{2}{3}$ and to the left with probability $\frac{1}{3}$. Recall that the Metropolis-Hastings algorithm creates a Markov chain (X_n) on some subset of $\mathbb{Z}$ by proposing a transition from i to j with probability Q_{ij} and accepting the proposal if $\frac{b_j Q_{ji}}{b_i Q_{ij}} \geq U$ where U is distributed uniformly in $[0,1]$. Let P_{ij} be the transition matrix for (X_n).

   (a) (3 points) What is the simplest choice of $b = (b_i)_{i\in\mathbb{Z}}$ so that (X_n) has long term probabilities
   $$\pi_i = \frac{1}{L}$$
   for $i = 1, 2, \ldots, L$ and is zero for all other states i? You do not have to justify your answer in this part.

   Solution: You can choose $b_i = C\pi_i$ where C is any constant, so the simple choice is $b_i = 1$ for $i = 1, \ldots, L$ and otherwise $b_i = 0$. (The fact that you can use any C is the reason why this algorithm works well on problems where the state space is huge).

   (b) (2 points) With the choice you made in part (a) what is $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n X_j^2$ when $L = 3$? What theorem is being used to obtain your answer?

   Solution: By the ergodic theorem proved in homework with $f(i) = i^2$,
   $$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n X_j^2 = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n f(X_j)$$
   $$= \sum_i \pi_i f(i) = \frac{1}{3}1^2 + \frac{1}{3}2^2 + \frac{1}{3}3^2 = \frac{14}{3}.$$

(c) (2 points) With the choice you made in part (a) find $b_1 P_{12}$ explicitly.

Solution: $b_1 Q_{12}\left(\frac{b_2 Q_{21}}{b_1 Q_{12}} \wedge 1\right) = \frac{2}{3}\left(\frac{\frac{1}{3}}{\frac{2}{3}} \wedge 1\right) = \frac{1}{3}$

(d) (2 points) With the choice you made in part (a) find $b_2 P_{21}$ explicitly

Solution: $b_2 Q_{21}\left(\frac{b_1 Q_{12}}{b_2 Q_{21}} \wedge 1\right) = \frac{1}{3}\left(\frac{\frac{2}{3}}{\frac{1}{3}} \wedge 1\right) = \frac{1}{3}$

(e) (1 point) With the choice you made in part (a) find P_{22} explicitly. Assume $L > 2$.

Solution: $P_{23} = Q_{23}\left(\frac{b_3 Q_{32}}{b_2 Q_{23}} \wedge 1\right) = \frac{2}{3}\left(\frac{\frac{1}{3}}{\frac{2}{3}} \wedge 1\right) = \frac{1}{3}$ and $P_{21} = \frac{1}{3}$ so $P_{22} = \frac{1}{3}$ so that the row of the probability transition matrix sums to one.

6. (a) (4 points) A Poisson process $(N(t), t \geq 0)$ of rate λ is a counting process defined by three properties. What are they? Write them out in detail if you can.

Solution:

1. $N(0) = 0$.
2. For $s_1 < s_2 \leq t_1 < t_2$, $N(t_2) - N(t_1)$ is independent of $N(s_2) - N(s_1)$.
3. For $s, t \geq 0$, $\mathbb{P}\big(N(s+t) - N(s) = n\big) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$.

(b) Local telephone calls arrive at a fundraiser according to a Poisson process with rate $\frac{18}{10}$ per minute. Long distance telephone calls arrive according to an independent Poisson process with rate $\frac{2}{10}$ per minute. In the following parts try to give answers in simple forms like $e^{-2} \times \left(\frac{6}{10}\right)^6$ instead of sums and integrals.

i. (2 points) What is the probability that no long distance call arrives between 2pm and $2:05$pm.

Solution: $\mathbb{P}(N_L(5) = 0) = e^{-2/10 \times 5} = e^{-1}$.

ii. (1 point) What is the probability that at least two calls arrive between 3pm and 3 : 03pm.

Solution: The sum of the two independent Poisson processes is a Poisson process $(N(t))$ with rate $\frac{18}{10} + \frac{2}{10} = 2$ calls per minute. $\mathbb{P}\big(N(2) \geq 2\big) = 1 - \mathbb{P}\big(N(2) = 0\big) - \mathbb{P}\big(N(2) = 1\big) = 1 - e^{-2\times 3} - \frac{6}{1!}e^{-2\times 3} = 1 - 7e^{-6}$.

iii. (2 points) If there are 5 calls in a five minute period, what is the probability that the first of these calls is in the first minute?

Solution: Easiest to calculate probability that the first call is not within the first minute. Recall that conditional on $N(5) = 5$ the times are uniform so we want the probability that all five calls are in the time interval $[1, 5]$ which is $\left(\frac{4}{5}\right)^5$. Therefore the answer is $1 - \left(\frac{4}{5}\right)^5$.

Alternative solution: rewrite the event that the first call is in the first minute as $\{N(1) \geq 1\}$. Then

$$\mathbb{P}\Big(N(1) \geq 1 | N(5) = 5\Big) = 1 - \mathbb{P}\Big(N(1) = 0 | N(5) = 5\Big)$$

$$= 1 - \frac{\mathbb{P}\Big(N(1) = 0, N(5) = 5\Big)}{\mathbb{P}\Big(N(5) = 5\Big)} = 1 - \frac{\mathbb{P}\Big(N(1) = 0, N(5) - N(1) = 5\Big)}{\mathbb{P}\Big(N(5) = 5\Big)}$$

$$= 1 - \frac{\mathbb{P}\Big(N(1) = 0\Big)\mathbb{P}\Big(N(5) - N(1) = 5\Big)}{\mathbb{P}\Big(N(5) = 5\Big)}$$

$$= 1 - \frac{\mathbb{P}\Big(N(1) = 0\Big)\mathbb{P}\Big(N(4) = 5\Big)}{\mathbb{P}\Big(N(5) = 5\Big)}$$

$$= 1 - \frac{e^{-2}\frac{8^5}{5!}e^{-8}}{\frac{10^5}{5!}e^{-10}} = 1 - \left(\frac{4}{5}\right)^5.$$

iv. (1 point) What is the probability that the first two calls are long distance.

Solution: The probability law of the local and long distance process is equivalent to independently randomly classifying the calls of the total process as "long distance" with probability $\frac{\frac{18}{10}}{\frac{18}{10} + \frac{2}{10}} = \frac{9}{10}$ and "local" with probability $\frac{1}{10}$. Therefore the probability that the first two calls are long distance is $\frac{1}{10} \times \frac{1}{10}$.

7. A small barbershop, operated by a single barber, has room for at most two customers including the one the barber is working on. Customers arrive according to a Poisson process of rate three per hour, but only enter if the shop is empty or has one customer, otherwise they go away. The barber takes an independent exponential random time with mean $\frac{1}{4}$ hour to cut the hair of each customer. Let $X(t)$ be the number of customers in the barber's shop. The possible values of $X(t)$ are $0, 1, 2$. Let $P(t) = (P_{ij}(t))$ be the probability transition function for this birth/death process.

   (a) (3 points) How is $P_{ij}(t)$ defined in terms of a conditional probability of $X(t)$?

   Solution: $P_{ij}(t) = \mathbb{P}\Big(X(t) = j | X(0) = i\Big)$.

   (b) (4 points) Find the derivative $P'(0)$ of $P(t)$ at $t = 0$ as an explicit 3×3 matrix, or equivalently, find v_i and q_{ij} for $i = 1, 2, 3$ and $j \neq i$.

   Solution: $P'(0) = \begin{bmatrix} -v_1 & 3 & 0 \\ 4 & -v_2 & 3 \\ 0 & 4 & -v_3 \end{bmatrix}$ where v_1, v_2, v_3 are such that the rows sum to zero, that is $v_1 = 3$, $v_2 = 7$, $v_3 = 4$.

(c) (2 points) If there is one customer in the shop at time zero what is the expected time until the first transition, that is, until there is either no customers or two customers in the shop?

> **Solution:** A continuous time MC in state i jumps for the first time at a random time $T^{(i)}$ with law $\exp\left[v_i\right]$ and this has mean $\frac{1}{v_i}$. For state $i = 1$ $v_1 = 7$ so the answer is $\frac{1}{7}$.
> To see this in more detail: The first transition out of state 1 takes place at a random time $T^{(1)}$ which is the minimum of the exponential(3) time for first arrival and the exponential(4) time for first departure. The minimum of these two independent exponential times is $\exp(3+4)$ and this has mean $\frac{1}{3+4}$ of an hour.

(d) (1 point) After the barber has been working for a long time what is the probability that a customer arrives, but has to leave because the barbershop is full?

> **Solution:** We have to find the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3)$ and then the answer is π_2. The equations of detailed balance are
>
> $$\pi_1 q_{10} = \pi_0 q_{01}, \qquad \pi_2 q_{21} = \pi_1 q_{12}, \qquad \pi_3 q_{32} = \pi_2 q_{23},$$
>
> From which
>
> $$\pi_1 = \pi_0 \frac{q_{01}}{q_{10}}, \qquad \pi_2 = \pi_0 \frac{q_{01}}{q_{10}} \frac{q_{12}}{q_{21}}, \qquad \pi_3 = \pi_0 \frac{q_{01}}{q_{10}} \frac{q_{12}}{q_{21}} \frac{q_{23}}{q_{32}}.$$
>
> Putting in the numbers from part (b),
>
> $$\pi_1 = \pi_0 r \qquad \pi_2 = \pi_0 r^2 \qquad \pi_3 = \pi_0 r^3$$
>
> where $r = \frac{3}{4}$. Determine π_0 so that $\pi_0 + \pi_1 + \pi_2 = 1$,
>
> $$\pi_0(1 + r + r^2) = 1.$$
>
> Therefore $\lim_{t\to\infty} \mathbb{P}(X_t = 2) = \pi_2 = \frac{r^2}{1+r+r^2} = \frac{9}{37}$.