University of British Columbia
Math 301, Section 201 (Froese)
Final Exam, April 2017

Name (print): ______________________________

Student ID Number: ________________ **Signature:** ______________________

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	15	
2	15	
3	20	
4	15	
5	10	
6	15	
7	10	
Total:	100	

Additional Instructions:

- No notes, books or calculators are allowed.
- Read the questions carefully and make sure you provide all the information that is asked for in the question.
- Show all your work. Correct answers without explanation or accompanying work could receive no credit.
- Answer the questions in the space provided. Continue on the back of the page if necessary.

1. (15 points) Compute the integral

$$I = \int_0^\infty \frac{x^{1/3}}{1+x^3} dx.$$

Sketch the contours used in the computation and relevant branch cuts. Indicate which terms tend to zero when taking a limit and provide the estimates that show this.

2. (15 points) Compute the integral

$$I = \int_0^\infty \frac{\cos(2x) - 1}{x^2} dx.$$

Sketch the contours used in the computation and indicate which terms tend to zero. You need not prove the estimates in this problem.

3. Consider the multivalued function

$$g(z) = (z^3 - 4z)^{1/3}.$$

(a) (5 points) Is $z = \infty$ a branch point for $g(z)$? Give a reason.

(b) (5 points) Use the range of angles method to construct a branch $G(z)$ of $g(z)$ that is analytic outside the circle $|z| = 2$ and positive for $z \in (2, \infty)$.

(c) (5 points) Compute $G(2i)$

(d) (5 points) Compute the value of $\displaystyle\oint_{|z|=3} \frac{1}{G(z)} dz$ where the contour is traversed once counterclockwise. (*Hint:* Find the residue at infinity.)

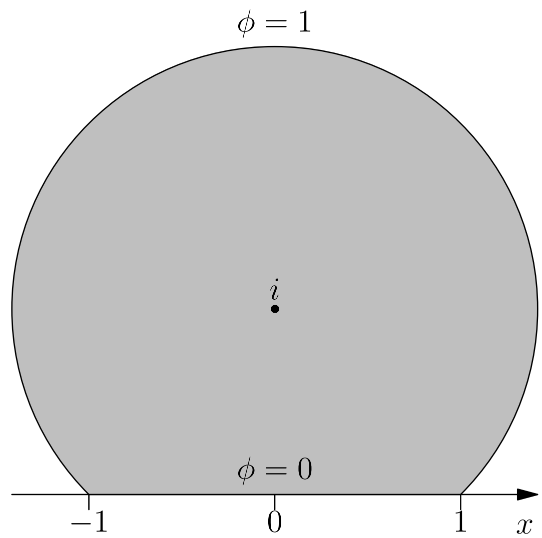


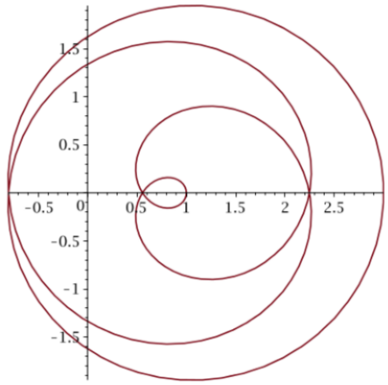
Figure 1: The region R

4. Let R be the part of the disk $|z - i| = \sqrt{2}$ lying above the real axis as shown:

   (a) (8 points) Find a fractional linear transformation $f(z)$ that maps R onto a sector of the form $\{z : 0 \leq \operatorname{Arg}(z) \leq \alpha\}$. What is α?

(b) (7 points) Find a harmonic function $\phi(x, y)$ defined for $(x, y) \in R$ satisfying the indicated boundary conditions.

5. The diagram shows the path traced out by the polynomial $p(z) = z^4 - z^3 + 1$ as z goes around the unit circle.



(a) (5 points) How many zeros does $p(z)$ have inside the unit circle?

(b) (5 points) Looking at the picture, how many zeros do you think that $\dfrac{1}{4}z^6 + p(z)$ has inside the unit circle? Explain your answer.

6. (15 points) Find a solution to

$$-u''(x) + u(x) = e^{-2a|x|}$$

for $a > 0$ using the Fourier transform.

7. Let $y(t)$ be the solution of the initial value problem

$$y''''(t) + 2y''(t) + y(t) = e^{-t}, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

(a) (5 points) Find the Laplace transform $Y(s)$ of $y(t)$.

(b) (5 points) Write a formula for the solution $y(t)$ as a sum of residues and explain why the solution includes polynomial $\times$ exponential terms like te^{it}.

Fourier Transform Summary

	$f(x)$	$\widehat{f}(k)$
Definition and inversion		
	$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\widehat{f}(k)dk$	$\int_{-\infty}^{\infty} e^{-ikx}f(x)dx$
Examples		
1	$\dfrac{1}{1+x^2}$	$\pi e^{-\lvert k\rvert}$
2	$e^{-\lvert x\rvert}$	$\dfrac{2}{1+k^2}$
3	$\begin{cases} 1 & \lvert x\rvert < 1 \\ 1/2 & x = 0 \\ 0 & \lvert x\rvert \geq 1 \end{cases}$	$2\,\dfrac{\sin(k)}{k}$
4	$\dfrac{\sin(x)}{x}$	$\pi \begin{cases} 1 & \lvert k\rvert < 1 \\ 1/2 & k = 0 \\ 0 & \lvert k\rvert > 1 \end{cases}$
5	$e^{-x^2/(2\sigma^2)}$	$\sqrt{2\pi}\,\sigma e^{-\sigma^2 k^2/2}$
Properties		
0	$c_1 f_1(x) + c_2 f_2(x)$	$c_1\widehat{f}_1(k) + c_2\widehat{f}_2(k)$
1	$f(x+a)$	$e^{iak}\widehat{f}(k)$
2	$e^{iax}f(x)$	$\widehat{f}(k-a)$
3	$f(ax)$	$a^{-1}\widehat{f}(k/a)$
4	$f'(x)$	$ik\widehat{f}(k)$
5	$ixf(x)$	$-\widehat{f}'(k)$
6	$f * g(x)$	$\widehat{f}(k)\widehat{g}(k)$
7	$f(x)g(x)$	$(2\pi)^{-1}\widehat{f} * \widehat{g}(k)$

Note about the examples 3 and 4: The value of a Fourier or inverse Fourier transform is insensitive to changes of the input function at a single point. In property 6: $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$

Laplace Transform Summary

	$y(t)$	$Y(s)$
Definition and inversion		
	$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}Y(s)ds$	$\int_0^{\infty} e^{-st}y(t)dt$
Examples		
1	e^{-at}	$\dfrac{1}{s+a}$
2	$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
3	$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
Properties		
0	$c_1 y_1(t) + c_2 y_2(t)$	$c_1 Y_1(s) + c_2 Y_2(s)$
1	$y'(t)$	$sY(s) - y(0)$
2	$ty(t)$	$-Y'(s)$
3	$e^{at}y(t)$	$Y(s-a)$
4	$u(t-a)y(t-a)$	$e^{-ac}Y(s)$
5	$y_1 * y_2(t)$	$Y_1(s)Y_2(s)$

In property 4: $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$. In property 5 $y_1 * y_2(t) = \int_0^t y_1(t-\tau)y_2(\tau)d\tau$