# The University of British Columbia
# Math 301 — Applied Complex Analysis
# 2014, April 30
### Omer Angel

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## Instructions

• **Explain** your reasoning thoroughly, and **justify** all answers (even if the question does not specifically say so).
• Calculators, but no other aides are permitted.
• If you need more space, additional paper is available. Always **note** which questions are solved in a separate booklet, or they may not be graded.
• Duration: **150** minutes.

Good luck, and enjoy the break.



| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 15 | |
| 3 | 20 | |
| 4 | 10 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 15 | |
| Total: | 100 | |

(10) 1. Compute $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+\pi^2}$. Justify all steps.

| $f(x)$ | $\hat{f}(k)$ | $f(t)$ | $F(s)$ |
|---|---|---|---|
| $\frac{1}{1+x^2}$ | $\pi e^{-\lvert k\rvert}$ | $e^{-at}$ | $\frac{1}{s+a}$ |
| $e^{-\lvert x\rvert}$ | $\frac{2}{1+k^2}$ | $\sin(at)$ | $\frac{a}{s^2+a^2}$ |
| $e^{-x^2/(2\sigma^2)}$ | $\sqrt{2\pi}\sigma e^{-\sigma^2 k^2/2}$ | $\cos(at)$ | $\frac{s}{s^2+a^2}$ |

(15) 2. Compute $\int_0^\infty \frac{x^{-2/3}}{x+3}$. State which branch and the contour you are using. Justify all steps.

(20) 3. Consider the branch of $(z^3 - 2z^2)^{1/3}$ given by $f(z) = \exp(\frac{2}{3}\operatorname{Log} z + \frac{1}{3}\operatorname{Log}(z-2))$, where $\operatorname{Log} z$ is the principle branch of $\log z$.

a. Where are the branch points of $(z^3 - 2z^2)^{1/3}$? Justify your claim, do not forget a possible branch point at $\infty$. Where are the branch cuts of $f$?

b. Express $f$ using the range of angles method.

c. Find the limit of $f(z)$ as $z$ approaches $1$ from above and from below.

d. Find the limit of $f(z)$ as $z$ approaches $-1$ from above and from below.

(10) 4. If $f(z) = \frac{P(z)}{Q(z)}$ is rational with $\deg(Q) \geq 2 + \deg(P)$, prove that the sum of the residue of $f$ at its poles is $0$.

(15) 5. a. Map the region $\{|z-2|>1\} \cap \{|z+2|>1\}$ to an annulus $\{a<|z|<b\}$ for some $a, b$.
b. Solve the Laplace equation $\Delta\varphi \equiv 0$ in that region with boundary conditions $\varphi = -1$ on the left circle and $\varphi = 1$ on the right circle. (Hint: $\log|z|$ is harmonic except at $0$.)

(15) 6. Let $f(x) = \begin{cases} xe^{-3x} & x > 0 \\ 0 & x < 0 \end{cases}$.

a. Compute the Fourier transform of $f$.

b. Compute the Laplace transform of $f$.

(15) 7. For a fixed number $a$, consider the solution of $y''(t) + ay'(t) + y(t) = \sin(3t)$ with $y(0) = 1$ and $y'(0) = 2$.

a. Find the Laplace transform $Y(s)$.

b. Determine for which values of $a$ the solution is bounded in $t$.