Math 301 Final Examination – April 24, 2010

THIS EXAM HAS 8 QUESTIONS.
YOU ARE PERMITTED ONE SHEET OF NOTES (DOUBLE SIDED).
NO CELLPHONES, BOOKS OR CALCULATORS.

1. (10pts) Use contour integration to calculate $\int_0^\infty \frac{\sin x}{x}\, dx$, including a brief explanation of every step.

2. (10 pts)

   (a) Construct a conformal map of the unit disk centred at the origin onto itself, that takes the point $i/2$ to the origin.

   (b) Construct a conformal map of the disk $|z-1| < 1$ onto the whole left half plane.

3. (10 pts) Solve Laplace's equation for ϕ in the lens-shaped region between the circles $|z-i| = 1$ and $|z-1| = 1$ with the boundary conditions that $\phi = 0$ on the circle $|z-i| = 1$ and $\phi = 1$ on the circle $|z-1| = 1$.

4. (15 pts)

   (a) Use residue calculus to verify the sum

   $$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \frac{\pi}{a} \coth(\pi a).$$

   (b) Use the result of part (a) to calculate $\sum_{k=1}^{\infty} \frac{1}{k^2}$, explaining all steps in full detail.

5. (15 pts) Let $f(z) = \frac{(1-z^2)^{1/2}}{1+z^2}$.

   (a) Find the residue of $f(z)$ at infinity.

   (b) Use an appropriate branch of $f(z)$ with a dogbone contour and residue calculus to show that

   $$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx = (\sqrt{2}-1)\pi.$$

(continued on page 2)

6. (15 pts) Find the inverse Laplace transform of

$$F(s) = \frac{\sinh xs^{1/2}}{s^2 \sinh s^{1/2}}$$

on $0 < x < 1$. No branch cut is needed here. You should describe the method in full detail but you can omit estimates of integrals.

7. (10 pts) Show that

$$\int_0^\infty e^{-is^2} ds = (1-i)\sqrt{\frac{\pi}{8}}.$$

8. (15 pts) Use the Fourier Transform to solve the Schroedinger equation

$$\begin{aligned} iU_t + U_{xx} &= 0 \qquad -\infty < x < \infty \qquad t > 0 \\ U(x,0) &= f(x) \end{aligned}$$

with $f(x) \to 0$ as $|x| \to \infty$. Your solution should be in the convolution form

$$U(x,t) = \int_{-\infty}^{\infty} f(x')g(x-x')dx'$$

where the function g must be explicitly determined. Show all work, but you may use the result of question 7 without proof in your solution.

(end of exam)