The University of British Columbia
Math 301 Final Examination - April 2009

1. [10] Find **all** values of:

(a)

$$(i^2)^i$$

(b)

$$(i^i)^2$$

(c)

$$\text{arcsinh}(i)$$

2. [18] Evaluate the integrals; justify all steps carefully
(a)

$$K = p.v \int_0^\infty \frac{x^{1/3}}{x^2 - 1} dx$$

(b)

$$J = \int_0^\infty \frac{x^{1/2} \log(x)}{1 + x^2} dx$$

3. [18]
(a) Carefully construct a branch of the function

$$g(z) = [z(1-z)]^{1/2}$$

with a branch cut on the interval $0 \le x \le 1$ such that $g(\frac{1}{2}(1+i)) = -\frac{1}{\sqrt{2}}$.

(b) Evaluate; justify all steps carefully

$$J = \int_0^1 x^{\frac{1}{2}} (1-x)^{\frac{1}{2}} dx.$$

4. [18] (a) Find the image of the upper half $z-$plane ($y \ge 0$) under the conformal mapping $s = f(z)$

$$s = \xi + i\eta = f(z) = \frac{z - i}{z + i}.$$

(b) Find the image of the **interior** of the unit circle in the $s-$ plane under the conformal mapping $w = g(s)$

$$w = u + iv = g(s) = \frac{s}{(1-s)^2}.$$

[Hint: first show that the image of the circle is real.]

(c) Simplify the combined mapping, defining $G(z)$

$$w = u + iv = g(f(z)) = G(z). \tag{1}$$

(d) What is the complex potential $F(z)$ of a uniform flow parallel to the $x-$ axis in the upper half $z-$ plane?
Use the above mapping to find the image of this uniform flow in the $w-$ plane defined by (1).
Represent the images parametrically as $u(t)$, $v(t)$, then eliminate v to give the streamlines in the form $u = H(v)$.
Sketch several curves.

5. [18]
(a) Find the Fourier transform, justify all steps carefully

$$g(x) = e^{-|x|}, \; -\infty < x < \infty.$$

(b) Use a Fourier transform to solve

$$u_{tt} + 2u_t + u = u_{xx}, \; -\infty < x < \infty, \; 0 < t,$$
$$u(x,0) = e^{-|x|}, \; u_t(x,0) = 0, \; -\infty < x < \infty.$$

Write the solution in the form

$$u(x,t) = \int_0^\infty G(x,t,\omega) d\omega.$$

6. [18]
(a) Find the inverse Laplace transform, justify all steps carefully

$$G(x,t) = L^{-1}\left(\frac{e^{-x\sqrt{s}}}{\sqrt{s}}\right)$$

(c) Use a Laplace transform to find the solution to the problem:

$$\begin{aligned} u_t &= u_{xx}, \; 0 < x < \infty, \; 0 < t, \\ u(x,0) &= 0, \; u(0,t) = \frac{1}{\sqrt{t}}, \; t > 0. \end{aligned}$$

You may use the results

$$\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}; \quad \int_0^\infty \frac{1}{\sqrt{u}} e^{-au} du = \sqrt{\frac{\pi}{a}}.$$