The University of British Columbia

Math 301 (201) Final Examination - April 2007

Closed book exam. No notes or calculators allowed
Answer all 5 questions. Time: 2.5 hours

1. [20]
The complex potential $w(z)$ for a source of strength 2π located at $z = a$ in a steady inviscid flow is

$$w(z) = \log(z - a)$$

A source of strength 2π is located at $z = 1 + i$, and the $x-$ axis is a solid barrier.
For this flow, find:
(a) The complex potential of this flow $w_1(z)$ in the region $\mathrm{Im}(z) > 0$.
(b) The velocity components along the $x-$ axis.
(c) The velocity components at the point $x = 0$, $y = \sqrt{2}$.

A second solid barrier is now introduced along the line $x = 0$.
(d) What is the complex potential of the flow, $w_2(z)$, in the first quadrant?
(e) Find the velocity components at the point $x = 0$, $y = \sqrt{2}$.

2. [20]
Evaluate, carefully explaining all steps,

$$J = \int_0^\infty \frac{(\mathrm{Log}\, x)^2}{1 + x^2} dx$$

Note that $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$.

3. [20]
(a) A function is defined as

$$g(z) = z^{\frac{1}{2}}(1 - z)^{\frac{1}{2}}$$

with a finite branch cut for $y = 0$, $0 < x < 1$, and $g(\frac{1+i}{2}) > 0$.
Find $g(i)$ and $g(\frac{1-i}{2})$.
(b) Evaluate, carefully explaining all steps,

$$I = \int_0^1 \frac{x dx}{x^{\frac{1}{2}}(1 - x)^{\frac{1}{2}}(4 + x)}.$$

4. [20]
(a) Solve:

$$\begin{aligned} \nabla^2\phi &= 0 \text{ in } \{\operatorname{Im} z > 0\} \cap \{|z-2i| > 2\} \\ \text{with} &: \quad \phi = 0 \text{ on } \operatorname{Im} z = 0 \text{ and } \phi = 5 \text{ on } |z-2i| = 2. \end{aligned}$$

What is $\phi(3,4)$?

5. [20]
(a) Find the Fourier transform, carefully explaining all steps, of

$$f(x) = \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Note that we can define the Fourier transform of $f(x)$ and its inverse by:

$$\begin{aligned} \mathcal{F}(f(x)) &= \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixk} dx, \\ \mathcal{F}^{-1}(\hat{f}(k)) &= f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ixk} dk. \end{aligned}$$

(b) Solve the boundary-value problem:

$$\begin{gathered} u_t - u_{xx} + u = 0, \quad -\infty < x < \infty, \ 0 < t; \\ u(x,0) = g(x), \quad -\infty < x < \infty. \end{gathered}$$

Here $g(x) \to 0$ as $|x| \to \infty$.
You may need the result:

$$\mathcal{F}^{-1}(e^{-\alpha k^2}) = \frac{1}{2\sqrt{\alpha}} e^{-x^2/4\alpha}.$$

The final solution can be left as a convolution integral.