The University of British Columbia

Math 301 (201) Final Examination - April 2005

Closed book exam. No notes or calculators allowed.
Answer all 5 questions. Time: 2.5 hours.

1. [15]
The flow field given by a source located at $z = 1$ is modified by the introduction of an infinite barrier at $x = 0$. For what values of y is the speed on the barrier k times the speed at the same location without the barrier? What is the possible range of k? Explain.

2. [30]
(a) Evaluate:

$$I = \int_0^\infty \frac{xdx}{8 + x^3}.$$

Hint: you should consider using a contour that includes the ray $\theta = \frac{2\pi}{3}$.

(b) By integrating around the finite branch cut $[-1, 1]$ (and using symmetry), evaluate

$$J = \int_0^1 \frac{x^4 dx}{(1 - x^2)^{\frac{1}{2}}(1 + x^2)}.$$

(c) Show by considering the two cases $x > 0$ and $x < 0$ that

$$p.v. \int_{-\infty}^\infty \frac{e^{i\omega x}}{\omega^2 - 1} d\omega = -\pi \sin |x|.$$

3. [20]
(a) Find a conformal mapping $w = f(z)$ that takes the region

$$\{|z - 1| < \sqrt{2}\} \cap \{|z + 1| < \sqrt{2}\}$$

into a portion of the right half plane.
Draw rough sketches of the regions in both the z and w planes.
It might be useful to check the image of $z = 0$.

(b) Find $\phi(x, y)$ that satisfies

$$\nabla^2 \phi = 0 \text{ in } \{|z - 1| < \sqrt{2}\} \ \cap \ \{|z + 1| < \sqrt{2}\}$$
$$\text{with : } \ \phi = 1 \text{ on } |z + 1| = \sqrt{2}, \text{ and } \phi = 2 \text{ on } |z - 1| = \sqrt{2}.$$

4. [20]
Let $f(x)$ and $g(x)$ be two absolutely integrable functions. Solve the boundary-value problem using Fourier transform, assuming $|u(x, y)|$ decays rapidly as $(x, y) \to \infty$.

$$u_{xx} + u_{yy} = f(x)e^{-y}, \ -\infty < x < \infty, \ 0 < y,$$
$$u(x, 0) = g(x), \ -\infty < x < \infty.$$

5. [15]
Solve the following ODE using Laplace transform and Bromwich formula:

$$y''' + y = 1, \quad (t > 0); \quad y(0) = y'(0) = 0, \quad y''(0) = 1.$$

Do not replace exponential functions by trigonometric functions in your solution.