Be sure that this examination has 12 pages including this cover

The University of British Columbia
Sessional Examinations - April 2012

Mathematics 300
Introduction to Complex Variables

Closed book examination   Time: $2\frac{1}{2}$ hours

Name ______________________ **Signature** ______________________

Student Number ______________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

No books, notes, or calculators are allowed.
Explain your reasoning carefully. You will be graded on the clarity of your explanations as well as on the correctness of your answers.

Rules Governing Formal Examinations

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
(a) having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
(b) speaking or communicating with other candidates; and
(c) purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		12
2		10
3		10
4		10
5		10
6		12
7		10
8		10
9		16
Total		100

Marks

[12] **1.** Express all of the following numbers in the form $a + ib$ with a and b real.

(a) $\sin\left[-i\log\left(1+\sqrt{3}\,i\right)\right]$

(b) $\mathrm{Log}\left[\frac{1-\mathrm{i}}{(1+\mathrm{i})^3}\right]$, where $\mathrm{Log}\,\mathrm{z}$ is the principal branch of $\log z$

(c) all solutions of $\sinh z = \frac{i}{\sqrt{2}}$

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[10] **2.** Let $u(x, y) = 2x^2 - 2y^2 - 3x + y$.

(a) Show that $u(x, y)$ is a harmonic function.

(b) Find all analytic functions $f(x+iy) = u(x, y)+iv(x, y)$ with $u(x, y) = 2x^2 - 2y^2 - 3x+y$ and $v(x, y)$ real.

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[10] **3.** Find a branch of $(1+z)^{1/2}$ which is analytic except for $z = x \geq -1$ and find its derivative at $z = -2$.

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[10] **4.** Evaluate the contour integral

$$\int_C \frac{dz}{(\bar{z}-1)^2}$$

where C is the semicircle $|z-1|=1$, $\operatorname{Im} z \geq 0$ from $z=0$ to $z=2$.

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[10] **5.** Find all entire functions $f(z)$ that obey "there is an integer n such that $|f(z)| < |z|^n + 1$ for all $\{\ z \in \mathbb{C}\ \big|\ |z| > 100\ \}$".

Continued on page 7

[12] **6.** Let

$$f(z) = \frac{1}{(2z-1)(z-2)}$$

(a) Expand $f(z)$ in a Laurent series valid in an annular region that contains $z = 1$. Give the region of convergence of your series.

(b) Evaluate $\oint_{C_{\pi/4}(0)} f(z)\, dz$ where $C_{\pi/4}(0)$ is the contour $|z| = \pi/4$ traversed once in the counterclockwise direction.

(c) Find the Taylor series of $f(z)$ about $z = 0$ and give its region of convergence.

(d) Compute $f^{(4)}(0)$.

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[10] **7.** (a) Show that

$$F(z) = \begin{cases} 1 & \text{if } z = 0 \\ \frac{e^z - 1}{z} & \text{if } z \neq 0 \end{cases}$$

is an entire function.

(b) Evaluate

$$\int_{C_2(0)} \frac{\cos z}{e^z - 1} dz$$

where $C_2(0)$ is the circle $|z| = 2$ traversed once in the counterclockwise direction.

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[10] **8.** Find the first four nonzero terms of the Taylor series at $z = 0$ for

$$\sin z \; \mathrm{Log}\,(1 - \mathrm{z})$$

where $\mathrm{Log}\,\mathrm{z}$ is the principal branch of $\log z$.

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[16] **9.** Evaluate the following definite integrals.

(a) $I_a = \displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)(x^2+4)}\,dx$

(b) $I_b = \displaystyle\int_{0}^{2\pi} \frac{d\theta}{1-2\alpha\cos\theta+\alpha^2}$ where the constant $0 < \alpha < 1$.

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The End