

Be sure this exam has 3 pages including the cover
The University of British Columbia
MATH 300, Section 101, Instructor Tai-Peng Tsai
Final Exam – December 2010

Family Name _____ **Given Name** _____

Student Number _____ **Signature** _____

No notes nor calculators.

Rules Governing Formal Examinations:

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
 - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
 - (b) Speaking or communicating with other candidates;
 - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers, and must not take any examination material from the examination room without permission of the invigilator; and
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

problem	max	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	15	
7.	15	
8.	10	
9.	10	
total	100	

1. Let $f(z) = e^{\frac{z+1}{z-1}}$. Find all z for which

(6 points) (a) $|f(z)| = 1$;

(4 points) (b) $|f(z)| < 1$.

(10 points) 2. Let $f(z) = 2x + 2y + i(x - y)^2$ where $z = x + iy$. Where is $f(z)$ differentiable in the complex plane? Where is $f(z)$ analytic? Explain your reasoning carefully.

(10 points) 3. Show that if v is a harmonic conjugate of u in a domain D , then $u^3 - 3uv^2$ is harmonic in D .

(10 points) 4. Find the radius of convergence of the Taylor series of $f(z) = \sqrt{2 - e^z}$ around $z = 1 + 4i$. Here the square root is given by the principle branch.

(10 points) 5. Let $f(z) = \frac{1}{z^2(z-3)}$. Find the maximum of $|f(z)|$ in the annulus $1 \leq |z| \leq 2$ and where it is attained.

6. All the circles below are oriented counterclockwise. Compute

(5 points) (a) $\int_{|z+1|=4} \bar{z}^2 dz$.

(5 points) (b) $\int_{|z|=100} z^2 \sin(z^{-1}) dz$.

(5 points) (c) $\int_{|z|=10} \frac{\sin(3z)}{(z+2)^{10}} dz$.

7. Let $f(z) = \frac{3}{(2z-1)(z-2)}$.

(5 points) (a) Give the first three nonzero terms for the Laurent series of $f(z)$ around $\frac{1}{2}$.

(5 points) (b) Give the first three nonzero terms for the Laurent series of $f(z)$ around 0, valid for small $|z|$. Give the region of convergence for the full expansion.

(5 points) (c) Give the first three nonzero terms for the Laurent series of $f(z)$ around 0, valid for large $|z|$. Give the region of convergence for the full expansion.

(10 points) 8. Find and classify isolated singularities of $f(z) = \frac{(\cos z - 1)e^{\frac{1}{z-1}}}{z^2(z+1)(z+2)^2}$.

(10 points) 9. Compute $\int_{|z|=2} \frac{e^{2z}}{z^2(z+3)} dz$. Here the circle $|z| = 2$ is oriented counterclockwise.

Some Formulas

1. **Cauchy's integral formula.** If f is analytic inside and on the simple closed positively oriented contour Γ and if z is any point inside Γ , then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad (n = 0, 1, 2, \dots).$$

2. **Laurent series.** If f is analytic in the annulus $0 \leq r < |z - z_0| < R$, then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta, \quad (k \in \mathbb{Z}),$$

where C is any positively oriented circle $|z - z_0| = \rho$, $r < \rho < R$.

3. If f has a pole of order m at z_0 , then

$$\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \left(\frac{d}{dz} \right)^{m-1} [(z - z_0)^m f(z)],$$

that is, the $(m-1)$ -th Taylor coefficient of $g(z) = (z - z_0)^m f(z)$.