Be sure this exam has 3 pages including the cover

The University of British Columbia

MATH 300, Section 101, Instructor Tai-Peng Tsai

Final Exam – December 2010

Family Name ____________________ **Given Name** ____________________

Student Number ____________________ **Signature** ____________________

No notes nor calculators.

problem	max	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	15	
7.	15	
8.	10	
9.	10	
total	100	

1. Let $f(z) = e^{\frac{z+1}{z-1}}$. Find all z for which

(6 points) (a) $|f(z)| = 1$;

(4 points) (b) $|f(z)| < 1$.

(10 points) 2. Let $f(z) = 2x + 2y + i(x-y)^2$ where $z = x + iy$. Where is $f(z)$ differentiable in the complex plane? Where is $f(z)$ analytic? Explain your reasoning carefully.

(10 points) 3. Show that if v is a harmonic conjugate of u in a domain D, then $u^3 - 3uv^2$ is harmonic in D.

(10 points) 4. Find the radius of convergence of the Taylor series of $f(z) = \sqrt{2 - e^z}$ around $z = 1 + 4i$. Here the square root is given by the principle branch.

(10 points) 5. Let $f(z) = \dfrac{1}{z^2(z-3)}$. Find the maximum of $|f(z)|$ in the annulus $1 \le |z| \le 2$ and where it is attained.

6. All the circles below are oriented counterclockwise. Compute

(5 points) (a) $\displaystyle\int_{|z+1|=4} \bar{z}^2 \, dz$.

(5 points) (b) $\displaystyle\int_{|z|=100} z^2 \sin(z^{-1}) \, dz$.

(5 points) (c) $\displaystyle\int_{|z|=10} \frac{\sin(3z)}{(z+2)^{10}} \, dz$.

7. Let $f(z) = \dfrac{3}{(2z-1)(z-2)}$.

(5 points) (a) Give the first three nonzero terms for the Laurent series of $f(z)$ around $\dfrac{1}{2}$.

(5 points) (b) Give the first three nonzero terms for the Laurent series of $f(z)$ around 0, valid for small $|z|$. Give the region of convergence for the full expansion.

(5 points) (c) Give the first three nonzero terms for the Laurent series of $f(z)$ around 0, valid for large $|z|$. Give the region of convergence for the full expansion.

(10 points) 8. Find and classify isolated singularities of $\quad f(z) = \dfrac{(\cos z - 1)e^{\frac{1}{z-1}}}{z^2(z+1)(z+2)^2}$.

(10 points) 9. Compute $\displaystyle\int_{|z|=2} \frac{e^{2z}}{z^2(z+3)} dz$. Here the circle $|z| = 2$ is oriented counterclockwise.

Some Formulas

1. **Cauchy's integral formula**. If f is analytic inside and on the simple closed positively oriented contour Γ and if z is any point inside Γ, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \qquad (n = 0, 1, 2, \ldots).$$

2. **Laurent series**. If f is analytic in the annulus $0 \le r < |z - z_0| < R$, then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta, \quad (k \in \mathbb{Z}),$$

where C is any positively oriented circle $|z - z_0| = \rho$, $r < \rho < R$.

3. If f has a pole of order m at z_0, then

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \left(\frac{d}{dz} \right)^{m-1} [(z - z_0)^m f(z)],$$

that is, the $(m-1)$-th Taylor coefficient of $g(z) = (z - z_0)^m f(z)$.