FINAL EXAM

Math 300 Section 101

December 8, 2009

Last Name: _____

First Name: _____

Student Number: _____

Signature: _____

The exam is worth a total of 100 points with duration 2.5 hours. No books, notes or calculators are allowed. Justify all answers, show all work and **explain your reasoning carefully**. You will be graded on the clarity of your explanations as well as the correctness of your answers.

Problem	Points	
1	10	
2	8	
3	12	
4	15	
5	10	
6	10	
7	10	
8	10	
9	15	
Total	100	

Problem 1 (10 points) Find all complex numbers z for which $\cos z = 2$. Write your answers in the form $x + iy$ where x and y are real.

Problem 2 (8 points) Find

$$\int_\Gamma |z|^2 \, dz,$$

where Γ is the contour consisting of the directed line segment from i to $1 + i$, followed by the directed line segment from $1 + i$ to 1.

Problem 3 (12 points) Describe exactly the set S of complex numbers z for which the power series

$$\sum_{n=2}^{\infty} n z^{n-1}$$

converges, and find its sum for all z in S.

Problem 4 (15 points) Define the function
$$f(z) = \frac{e^z \sin z}{z^3}.$$
(a) Find three nonzero terms of the Laurent series of $f(z)$ in the domain $\{z \ : \ z \neq 0\}$.

(b) What kind of singularity does $f(z)$ have at $z = 0$?

(c) Find
$$\int_C f(z) \, dz,$$
where C is the positively oriented circle $\{z \ : \ |z| = 1\}$.

Problem 5 (10 points) Find all possible values of the integral

$$\int_\Gamma \frac{z+3}{z(z-3)}\,dz$$

for simple closed contours Γ where 0 and 3 are not on Γ.

Problem 6 (10 points) If $f(z)$ and $g(z)$ are functions analytic at $z = 0$, and $g'(0) \neq 0$, then what type of singularity does the function

$$h(z) = \frac{f(z) - f(0)}{g(z) - g(0)}$$

have at $z = 0$?

Problem 7 (10 points) Suppose that f is an analytic function in a domain D, with real and imaginary parts $u(z)$ and $v(z)$, and that $g(t)$ is a real function of one real variable t which is differentiable for all t. If $v(z) = g(u(z))$ for all z in D, then show that f is constant in D.

Problem 8 (10 points) Suppose that $f(z)$ is analytic everywhere in $\mathbb{C}$ except $z = 1$, and that

$$\int_C f(z) \, dz = 6\pi i,$$

where C is the positively oriented circle $\{z \; : \; |z - 1| = 2\}$. Find

$$\int_C z^2 f''(z) \, dz. \quad \text{(Hint: Use residues!)}$$

Problem 9 (15 points) (a) Use residue theory to find

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x}{(x^2 + 2x + 5)^2}\, dx.$$

(b) Why can't the expanding semicircular contour method as presented in class and the textbook be used to evaluate

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x^3}{(x^2 + 2x + 5)^2}\, dx?$$

Be detailed; specify what aspect of the method fails for this function.