Marks

[15] **1.** (a) An elastic string of length 4 with fixed ends has an initial shape $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 3 \\ 0 & \text{if } 3 < x \le 4 \end{cases}$$

It is released from rest at time $t = 0$. Assume that the displacement $u(x,t)$ satisfies

$$u_{xx} = u_{tt}, \qquad 0 \le x \le 4,\ t > 0.$$

Find $u(x,t)$.

(b) Sketch $u(x,0)$ and $u(x,1)$.

Continued on page 3

(a)

Continued on page 4

[20] **2.** Let $g(x) = -x$ be defined for $0 \leq x \leq 1$.

(a) Extend $g(x)$ as a periodic function of period 1. Find the Fourier series for $g(x)$ in complex form.

(b) Extend $g(x)$ as an odd function of period 2. Find the Fourier series for $g(x)$ in terms of sines and cosines.

(c) One end ($x = 0$) of a copper bar ($\alpha^2 = 1$) of length 1 is maintained at 0°C while the other is at 10°C. Initially the entire bar is at 0°C. Find the temperature $u(x,t)$ in the bar if $u(x,t)$ satisfies

$$\begin{aligned} u_t &= u_{xx} \quad && 0 \leq x \leq 1,\ t > 0 \\ u(0,t) &= 0 && t > 0 \\ u(1,t) &= 10 && t > 0 \end{aligned}$$

Continued on page 5

Continued on page 6

(a)

(b)

(c)

Continued on page 7

[15] **3.** (a) By direct integration, find the Fourier Transform of the function

$$a(t) = \mathrm{rect}\left(\tfrac{t}{2}\right)\cos(\pi t).$$

(b) Find the Fourier Transform of the function

$$b(t) = \begin{cases} \cos(t) & \text{if } \pi < t < 3\pi \\ 0 & \text{elsewhere} \end{cases}$$

(c) Find the Inverse Fourier Transform of

$$\widehat{c}(\omega) = \frac{4}{2 + 3i\omega - \omega^2}$$

Continued on page 8

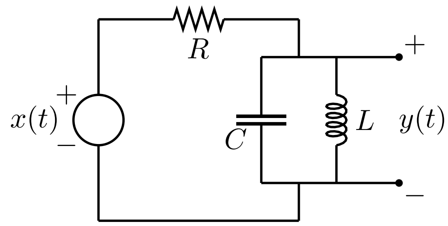
(a)

(b)

(c)

Continued on page 9

[15] **4.** In this problem you will analyze this circuit:



The input signal is a time-varying voltage $x(t)$ and the output signal is the voltage $y(t)$ measured across the inductor. Low-frequency signals face little opposition to flow through the inductor, so they get dissipated mostly by the resistor. High-frequency signals flow easily through the capacitor, so they also get dissipated by the resistor. But signals of some intermediate frequency are opposed by both reactive components, and produce large-amplitude outputs. The signals described above are related by the constant coefficient differential equation

$$RLCy''(t) + Ly'(t) + Ry(t) = Lx'(t).$$

(a) Let $\widehat{x}(\omega)$ and $\widehat{y}(\omega)$ be the Fourier transforms of $x(t)$ and $y(t)$. Define

$$H(\omega) = \frac{\widehat{y}(\omega)}{\widehat{x}(\omega)}, \qquad A(\omega) = |H(\omega)|, \qquad H(\omega) = A(\omega)e^{i\phi(\omega)}.$$

Find simple algebraic expressions for $H(\omega)$, $A(\omega)$ and $\tan(\phi(\omega))$.

(b) Use calculus to find the value of $\omega > 0$ at which $A(\omega)$ is maximized. This is the circuit's resonant frequency. Express your answer in terms of L, R, and C. [Hint: Maximize $|A(\omega)|^2$.]

Continued on page 10

(a)

(b)

Continued on page 11

[15] **5.** Consider the discrete time signal

$$x[n] = \sin \tfrac{\pi n}{2} \cos(\pi n)$$

(a) Is $x[n]$ periodic? If so, find a period N.

(b) Is the discrete Fourier transform $\widehat{x}[k]$ of this signal periodic? If so, find a period for $\widehat{x}[k]$.

(c) Find the discrete Fourier transform $\widehat{x}[k]$ of this signal.

Continued on page 12

(a)

(b)

(c)

Continued on page 13

[20] **6.** Consider an LTI system for which

$$y[n] + \tfrac{1}{4}y[n-1] - \tfrac{1}{8}y[n-2] = x[n]$$

(a) Use $X(z)$ and $Y(z)$ to denote the z–transforms of $x[n]$ and $y[n]$, respectively. Express the z-transform of $y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}$ in terms of $Y(z)$.

(b) Find the system function $H(z) = \frac{Y(z)}{X(z)}$ for this system.

(c) Plot the poles and zeroes of $H(z)$ and indicate the region of convergence, assuming that the system is causal.

(d) Using z–transforms, determine $y[n]$ if

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

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(a)

(b)

(c)

(d)

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The University of British Columbia
Final Examinations - April, 2007

Mathematics 267
Mathematical Methods for Electrical and Computer Engineering

Closed book examination Time: $2\frac{1}{2}$ hours

Name ______________________ **Signature** ______________________

Student Number ______________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

To receive full credit, all answers must be supported by clear and correct derivations.
No calculators, notes, or other aids are allowed. A formula sheet is provided with the exam.
Use the backs of the sheets, if necessary, for additional work. But please write your final answers in the boxes provided.

Rules Governing Formal Examinations

1. Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

3. No candidate shall be permitted to enter the examination room after the expiration of one half hour from the scheduled starting time, or to leave during the first half hour of the examination.

4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

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2		20
3		15
4		15
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Total		100