The University of British Columbia
Final Examination -December 17, 2010
Mathematics 265

Instructor: Dr. Keshet

Closed book examination Time: 2.5 hours

LAST Name:____________________ **FIRST Name:**____________________

Student #: ____________ **Signature** ________________ **Section:** 101 / 103 / **(Circle one)**

Special Instructions: - Be sure that this examination has 13 pages. Write your *full* name (as on your Student ID) on top of each page. Circle your section number (MW 8:00AM is the 101 section. MW 9:00AM is the 103 section).

- No calculators, electronic devices, books, or notes are permitted. **The last page contains a table of Laplace transforms.** You can tear out that page for convenience.

- Unless otherwise indicated, show all your work. Answers not supported by calculations or reasoning may not receive credit. Messy work will not be graded.

- At the end of the examination period: 1. You will be instructed to put away all writing implements (Continuing to write past this signal is considered cheating). 2. Remain in your seats until exams have been collected. 3. You will be instructed when you are free to leave.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

  (a)having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;

  (b) speaking or communicating with other candidates; and

  (c) purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		24
2		18
3		10
4		12
5		12
6		12
7		12
Total		100

Problem 1: Multiple Choice Questions: Circle ONE correct answer (a, b, c, d, or e). There is no partial credit in this question.

1: Consider the (discontinuous) function described by

$$f(t) = \begin{cases} 0 & t < 1 \\ -(2t+1) & 1 \le t \le 2 \\ t^2 & 2 < t \le 4 \\ 0 & 4 < t \end{cases}$$

This function can be written in terms of step functions as follows:

(a) $f(t) = -(u_1(t) + u_2(t))(2t+1) + (u_2(t) + u_4(t))t^2$

(b) $f(t) = -(2t+1)u_1(t) + (t+1)^2 u_2(t) - t^2 u_4(t)$

(c) $f(t) = -(2t+1)u_1(t) + t^2 u_2(t)$

(d) $f(t) = (2t+1)u_2(t) + t^2 u_4(t)$

(e) $f(t) = -(2t+1)u_2(t) - t^2 u_4(t)$

2: Suppose that two solutions of a differential equation $y'' + p(t)y' + q(t)y = 0$ are $y_1(t) = t^2 - 2t + 1$ and $y_2 = t - 1$. Then the Wronskian of these solutions, W is

(a) $W = -(t-1)^2$

(b) $W = 3t^2 - 6t + 3$

(c) $W = (t-1)(2t-3)$

(d) $W = -t^2 + 6t - 1$

(e) $W = (t-1)^3$

3: To solve the ODE $y'' - y' - 2y = t + te^{-t}$, the form of the particular solution that is needed is

(a) $y_p(t) = At + Bte^{-t}$

(b) $y_p(t) = At + B + Cte^{-t}$

(c) $y_p(t) = At + B + (Ct + D)e^{-t}$

(d) $y_p(t) = At + B + t(Ct + D)e^{-t}$

(e) $y_p(t) = At^2 + t^2(Ct + D)e^{-t}$

4: Suppose that the current $I(t)$ in an LRC circuit that satisfies the ODE

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = f(t) = 4\cos(\omega_d t).$$

Suppose that the inductance is $L = 10$, the capacitance is $C = 0.1$. The circuit is tested with various resistors and input functions $f(t)$ of several different driving frequencies ω_d. For which of the following settings would this circuit produce oscillations with the greatest amplitude?

(a) $R = 20, \omega_d = 2$ (b) $R = 10, \omega_d = \sqrt{32}$ (c) $R = 1, \omega_d = 2$ (d) $R = 0.1, \omega_d = 11$ (e) $R = 0.1, \omega_d = 1.1$

5: Which of the following functions is the inverse Laplace transform of $F(s) = \dfrac{e^{-2s}}{s^2+4}$?

(a) $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}u_2(t)\sin(2t-4)$

(b) $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{4}u_2(t)\sin(2t-4)$

(c) $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}u_2(t)\sin(2t-2)$

(d) $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-2t-4}\sin(2t-2)$

(e) $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-2t}\sin(2t)$

6: Which of the following answers corresponds to the convolution of the functions $f(t) = e^t$ and $g(t) = e^{-2t}$?

(a) $e^{-t} - e^{-2t}$ (b) $\frac{1}{3}[e^t - e^{-2t}]$ (c) $-e^{-3t} + e^{-2t}$ (d) $e^t e^{-3\tau}$ (e) $e^{2t} - e^t$

Problem 2: Let $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, and consider the following systems of equations.

(a) Match each system with a corresponding phase plane diagram (circle **1** correct response: a,b,c,d, or None).

(b) For those cases that matched, draw a few arrows directly on the diagrams to indicate the direction of increasing time along the solution curves in the xy plane. (Hint: you do not need to solve the ODEs fully to figure out those directions.)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$ a b c d None

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix} \mathbf{x}$$ a b c d None

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x}$$ a b c d None

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 10 & 2 \\ -8 & 2 \end{bmatrix} \mathbf{x}$$ a b c d None

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \mathbf{x}$$ a b c d None

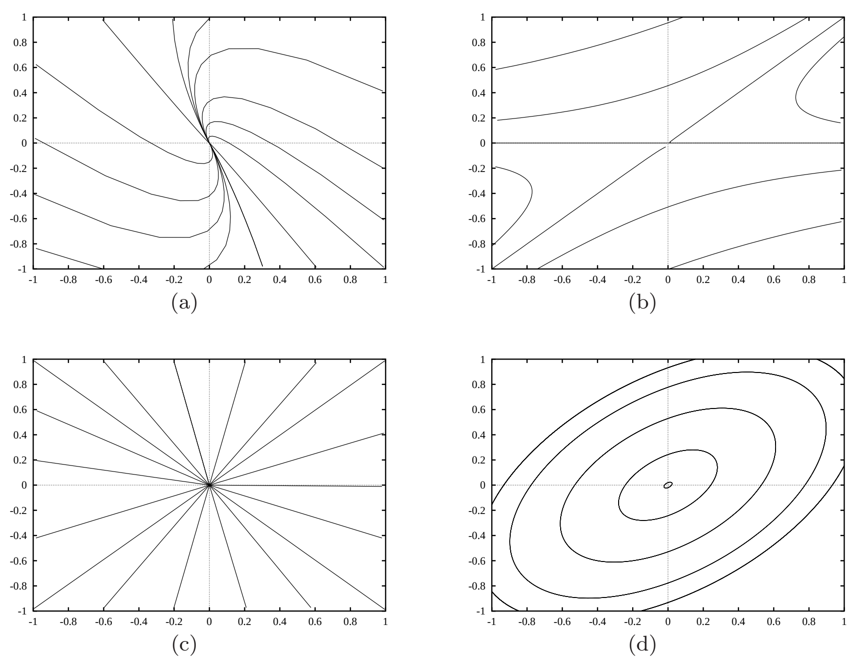

Figure 1: Phase plane plots for Problem 2. (x is the horizontal axis, y the vertical axis in each case)

Problem 3: Solve the following initial value problem:

$$\frac{dy}{dx} + xy = x, \quad y(0) = 2$$

Problem 4: Use the method of undetermined coefficients to solve the initial value problem

$$y'' - 6y' + 10y = 6\sin(2t), \quad y(0) = \frac{2}{5}, y'(0) = 0$$

Extra space (if needed)

Problem 5:

(a) Solve the initial value problem $y'' - 3y' + 2y = \delta(t-1), \quad y(0) = 1, y'(0) = 1.$
(b) What is the value of y at time $t = 2$?

Extra space (if needed)

Problem 6:

Solve the system of first order ODEs given below:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \mathbf{x}, \quad \text{with} \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Problem 7: For a holiday dinner, a large roast is to be cooked. At time $t = 0$, the roast is taken out of the refrigerator, and its initial temperature is $T(0) = 0°$ Celsius. It is left at room temperature ($E_{room} = 20°$ Celsius) for 1 hour. Then it is put into an oven ($E_{oven} = 220°$ Celsius) for 1 hour. After this time, it is left at room temperature until dinner.

Assume that Newton's Law of Cooling is a good approximation so that the temperature of the roast $T(t)$ at time $t > 0$ satisfies

$$\frac{dT}{dt} = k(E(t) - T), \quad \text{where the ambient temperature is} \quad E(t) = \begin{cases} E_{room} & 0 \le t \le 1 \\ E_{oven} & 1 < t \le 2 \\ E_{room} & 2 < t \end{cases}$$

For simplicity, assume that $k = 1$. Find the temperature of the roast, $T(t)$ for $t > 0$.

Extra space (if needed)

You may tear out this page for convenience. You do not need to submit it with the exam.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$\delta(t-c)$	e^{-cs}
$e^{ct}f(t)$	$F(s-c)$
$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$