

The University of British Columbia

Final Examination - December 2009

Mathematics 265

Section 101

Closed book examination

Time: 2.5 hours

Last Name: _____ First: _____ Signature _____

Student Number _____

Special Instructions:

- Be sure that this examination has 13 pages. Write your name on top of each page.
- You are allowed to bring into the exam one $8\frac{1}{2} \times 11$ formula sheet filled on both sides. No calculators or any other aids are allowed.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
 - (a) having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
 - (b) speaking or communicating with other candidates; and
 - (c) purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		15
5		20
6		20
7		15
Total		100

Some Laplace transforms:

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, s > 0$
t^n	$\frac{n!}{s^{n+1}}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}, s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, s > a $
$\cosh(at)$	$\frac{s}{s^2-a^2}, s > a $
$u_c(t)$	$\frac{e^{-cs}}{s}, s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$

[10] 1. Solve the initial value problem:

$$(e^x \sin x)y' = -1 - (e^x \cos x)y,$$

with $y(\pi/2) = e^{-\pi/2}$.

[10] **2.** Find all solutions of the differential equation

$$(x + 1)^3 y' + (x + 1)e^{-y} = 0.$$

[10] **3.** Consider the differential equation

$$y'' + p(t)y' + q(t)y = 0 \quad (*)$$

where p and q are continuous functions for all t .

(a) Can $y(t) = \sin(t^2)$ be a solution on an interval containing $t = 0$ of the differential equation $(*)$? Explain your answer.

(b) Calculate the Wronskian of t and t^2 .

(c) Can t and t^2 both be solutions of the same differential equation $(*)$? Explain clearly.

[15] 4. Use the method of undetermined coefficients to find the general solution of

$$y'' + y = \cos t.$$

Extra space (if needed)

[20] 5. Consider the initial value problem

$$y'' - 3y' + 2y = g(t), \quad y(0) = 1, \quad y'(0) = 1$$

where

$$g(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t \geq 1. \end{cases}$$

- (a) Compute the Laplace transform $\mathcal{L}\{g(t)\}$.
- (b) Use the method of Laplace transforms to solve the initial value problem.

Extra space (if needed)

[20] **6.** Consider the following 2×2 matrix with real coefficients

$$A = \begin{pmatrix} 3 & 1 \\ 0 & a \end{pmatrix},$$

and consider the system of differential equations

$$X'(t) = AX(t), \tag{1}$$

where $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

- (a) Write the determinant $|A - \lambda I|$ in factored form and then find the eigenvalues λ_1 and λ_2 of the matrix A in terms of a .
- (b) In the case where $a \neq 3$, find the eigenvectors corresponding to λ_1 and λ_2 .
- (c) If $a \neq 3$, find two solutions $X^{(1)}(t)$ and $X^{(2)}(t)$, in terms of a , so that $\{X^{(1)}, X^{(2)}\}$ forms a fundamental set of solutions for (1).
- (d) Deduce the general solution of (1) in the case $a \neq 3$.
- (e) What is/are the eigenvalue(s) of A in the case where $a = 3$? Find, in this case, fundamental solutions $X^{(1)}(t)$ and $X^{(2)}(t)$, and then the general solution of (1).

Extra space (if needed)

[15] 7. Consider the differential equation $y'' - 4y' + 13y = 0$.

- (a) Transform the above equation into a system of first order differential equations, and write it in matrix form $X'(t) = AX(t)$.
- (b) Find two real-valued solutions $X^{(1)}(t)$ and $X^{(2)}(t)$ that form a fundamental set of solutions to the system $X' = AX$ from part (a). What is the general solution of the system?
- (c) Describe the behaviour of the solutions as $t \rightarrow \infty$.

Extra space (if needed)