Math 265, Fall 2008
Final Exam, December 5th

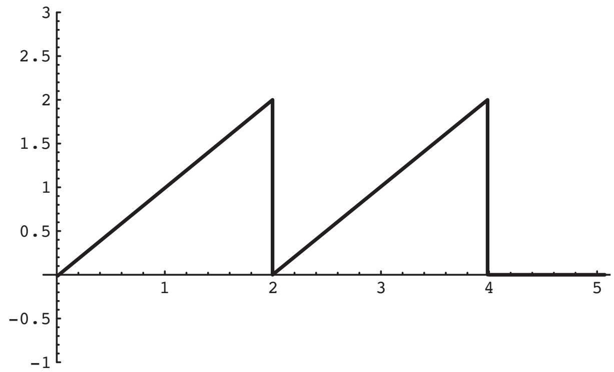
Instructions

- Answer all problems in the solution booklet. Work done on the question book will not be graded.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- No calculators or other aids, or notes or textbooks are allowed.
- Write your name and student number on every solution booklet you use.
- Put your student ID on your desk so it can easily be inspected.
- This exam has 5 problems. Answer all problems.
- There is a table of Laplace transforms on the last page.

Do not open this question book until you are told to do so.

1. *[50 POINTS]* This problem has ten parts, each worth five points.

i. Find the Laplace transform of the function shown in the graph.



ii. Solve the initial value problem for $y(t)$. a is a positive constant.

$$y' + \frac{2}{a}y = t \qquad y(0) = 1.$$

iii. Solve the initial value problem for $y(t)$ and draw a clear graph of the solution, indicating the period and amplitude.

$$y'' + 6y = 0 \qquad y(0) = 1,\ y'(0) = 1.$$

iv. Solve the initial value problem for $y(t)$ and draw a clear graph of the solution, indicating the period and amplitude.

$$y'' + 6y = \delta(t-1) \qquad y(0) = 0,\ y'(0) = 0.$$

v. Solve the initial value problem for $\vec{x}(t)$.

$$\vec{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \vec{x} \qquad \vec{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

vi. Give an example of a 2×2 matrix A such that the system $\vec{x}' = A\vec{x}$ has a centre point at $(0,0)$.

vii. An object of mass m is dropped from a height h. Let the velocity of the object at time t be $v(t)$ with $v(0) = 0$. The only forces acting on the object are due to gravity ($F_G = mg$) and drag ($F_D = kv$) where the drag coefficient, k, is a positive constant.

(i) In terms of meters, seconds, and kilograms, what are the units of k?
(ii) Find the terminal velocity of the object.
(iii) Find the distance the object has travelled after time t.

viii. Find the inverse Laplace transform of

$$F(s) = \frac{2s+1}{s^2+s+5}.$$

ix. A spherical raindrop evaporates at a rate proportional to its surface area. Initially, the radius of the raindrop is 3mm. After one minute, the radius of the raindrop is 2mm. At what time does the raindrop completely disappear?

x. Find the general solution of the differential equation. $f(t)$ is a unknown function of time.

$$y'(t) + f(t) = y(t).$$

Hint: your solution will be in terms of an integral.

2. *[12 POINTS]* Consider the forced LCR circuit, which can be described by:

$$Q''(t) + RQ'(t) + \frac{1}{C}Q = F\cos(\omega t), \quad Q(0) = 0, \quad Q'(0) = 1,$$

where Q(t) is the charge at time t, R the resistance, C the capacitance, and the inductance $L = 1$. The forcing has amplitude F and frequency ω.

On the following page you will find plots of different solutions $Q(t)$ for various values of R, C, F, and ω, corresponding in no particular order to:

A. $R = 0$, $C = 1/64$, $F = 1$, $\omega = 8 \Rightarrow$
$Q''(t) + 64Q(t) = \cos(8t), \quad Q(0) = 0, \quad Q'(0) = 1.$

B. $R = 0.8$, $C = 1/64$, $F = 0$, ω not relevant $\Rightarrow$
$Q''(t) + 0.8Q'(t) + 64Q(t) = 0, \quad Q(0) = 0, \quad Q'(0) = 1.$

C. $R = 0.8$, $C = 1/64$, $F = 1$, $\omega = 8 \Rightarrow$
$Q''(t) + 0.8Q'(t) + 64Q(t) = \cos(8t), \quad Q(0) = 0, \quad Q'(0) = 1.$

D. $R = 17$, $C = 1/64$, $F = 0$, ω not relevant $\Rightarrow$
$Q''(t) + 17Q'(t) + 64Q(t) = 0, \quad Q(0) = 0, \quad Q'(0) = 1.$

E. $R = 0$, $C = 1/64$, $F = 0$, ω not relevant $\Rightarrow$
$Q''(t) + 64Q(t) = 0, \quad Q(0) = 0, \quad Q'(0) = 1.$

F. $R = 0$, $C = 1/64$, $F = 1$, $\omega = 7 \Rightarrow$
$Q''(t) + 64Q(t) = \cos(7t), \quad Q(0) = 0, \quad Q'(0) = 1.$

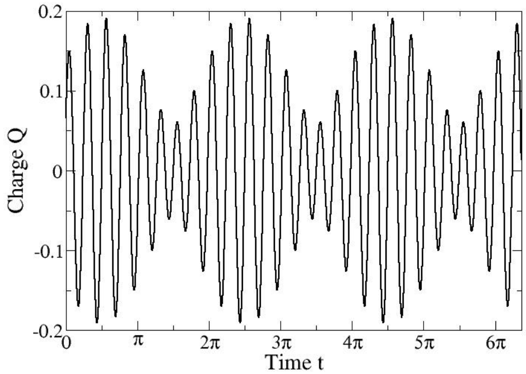
In your solution booklet, draw the following table and complete the entries to match the equations and their graphs:

Equation	Graph
A	
B	
C	
D	
E	
F	

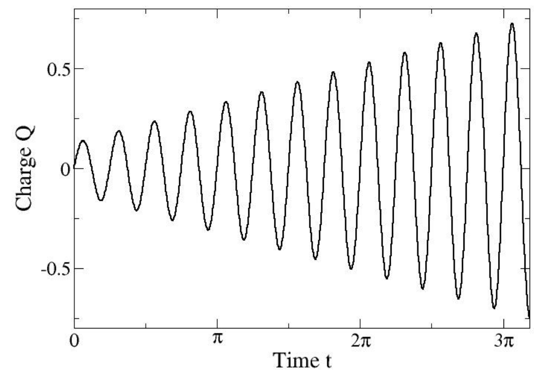
Hints:

1. You don't need to solve each problem completely.
2. Look at the plots carefully and don't forget to check the axis scales.

Problem 2, continued



1.



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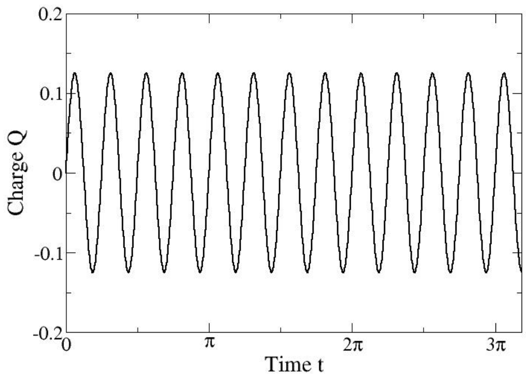

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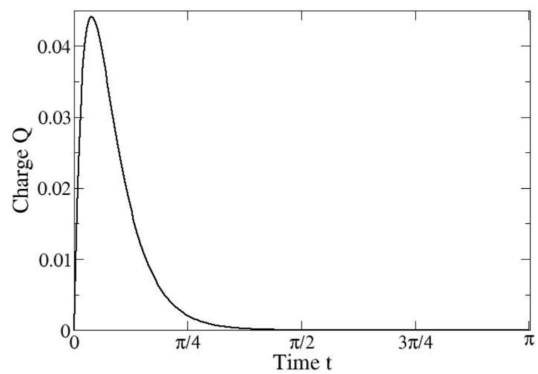

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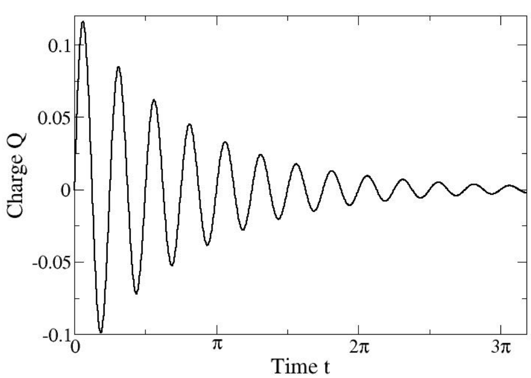

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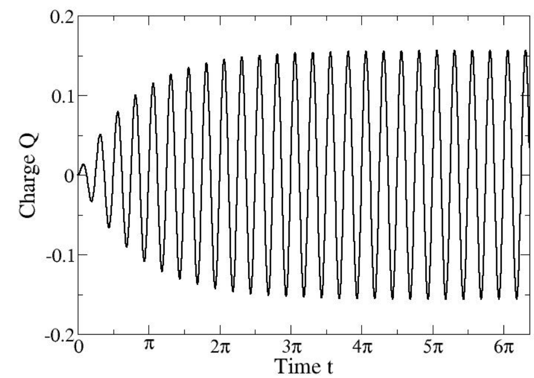


6.

3. *[13 POINTS]* Consider a servomechanism that models an automatic pilot. Such a mechanism applies torque to the steering control shaft so that a plane or boat will follow the prescribed course. If we let $y(t)$ be the true direction (i.e. the angle) of the craft at time t and $g(t)$ be the desired direction at time t, then

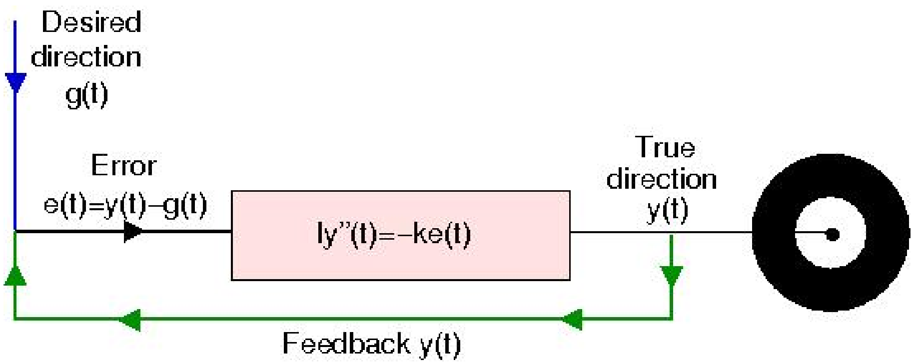
$$e(t) = y(t) - g(t)$$

denotes the **error** or **deviation** between the desired direction and the true direction.

Assume that the servomechanism can measure the error $e(t)$ and feed back to the steering shaft a component of torque that is proportional to $e(t)$, but opposite in sign (see figure). The rotational analogue of Newton's second law states that: (moment of inertia I)$\times$(angular acceleration)=(total torque). For our servomechanism this becomes:

$$Iy''(t) = -ke(t),$$

where k is a positive proportionality constant.



Using the method of Laplace transforms, determine the error $e(t)$ for the automatic pilot if the steering shaft is initially at rest in the zero direction ($y(0) = 0$, $y'(0) = 0$) and the desired direction is given by $g(t) = t$.

4. *[13 POINTS]* Consider

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -4 & -a \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

a is an unknown constant.

(a) Solve this system with $a = 3$.
(b) Solve this system with $a = -5$.
(c) Find all possible values of a for which (0,0) is an unstable spiral point.

5. *[12 POINTS]* Consider the following problem with ω a positive constant.

$$y'' + 9y = 4\cos\omega t \qquad y(0) = y'(0) = 0.$$

(a) Solve this problem assuming that $\omega \neq 3$.
(b) Solve this problem when $\omega = 3$.
(c) Draw a graph of the solution to the problem when $\omega = 2.9$. HINT: Rewrite your solution in a different form using $\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$.

END OF EXAM