The University of British Columbia

Final Examination - December 7, 2005

Mathematics 265

All Sections

Closed book examination | Time: 2.5 hours

Special Instructions:

- Be sure that this examination booklet has 4 pages.
- No calculators or notes other than your one page formula sheet are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

1. [10] Consider the ODE $y' = y(2-y)$.
   (a) Draw a direction field for this ODE.
   (b) Without solving the equation, determine $\lim_{t\to\infty} y(t)$ if $y(t)$ is a solution of this ODE with
      i. $y(0) = 0$,
      ii. $y(0) = 2$,
      iii. $y(0) = -1$.

2. [10] Consider the ODE
$$(t-1)\frac{dy}{dt} = \frac{2t}{t+1} + 1,$$
Find the solution of this differential equation that satisfies $y(0) = 0$. Determine the largest interval in which your solution is valid.

3. [20] Consider the initial value problem
$$y'' + 4y = g(t) \qquad y(0) = 3, \qquad y'(0) = 1.$$
   (a) Solve the corresponding homogeneous equation, and prove that the two solutions you have found are a fundamental set of solutions.
   (b) For each of the following cases, what form would you guess for a particular solution $Y(t)$, if you were going to use the method of undetermined coefficients? **Do not solve the equation!**
      i. $g(t) = t^2e^t$
      ii. $g(t) = e^t \sin t + t$
      iii. $g(t) = \cos(2t)$
   (c) Solve the initial value problem when $g(t) = 5\cos(t)$.

4. [15] Let $f(t) = \sin(t)$ and $g(t) = e^{-2t}$.

   (a) Compute $W(f, g)(t)$.

   (b) Are f and g linearly independent on $(0, 2\pi)$? Justify your answer.

   (c) Can one find an ODE $y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on $(0, 2\pi)$ for which both f and g are solutions? Explain.

   (d) Compute $h(t) = f \star g$ where "$\star$" denotes convolution.

5. [15] Solve the initial value problem

$$y'' + 3y' + 2y = (t-2)u_2(t), \quad y(0) = y'(0) = 0.$$

   What is $\lim_{t\to\infty} y(t)$?

6. [15] Consider the system of first-order ODEs given by

$$\begin{aligned} x_1' &= -3x_1 + 4x_2 + 4e^{-t} \\ x_2' &= -x_1 + 2x_2 + 3e^{-t} \end{aligned}$$

   (a) Write this system as a single matrix equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$.

   (b) Find the general solution of the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

   (c) Sketch the trajectory on the x_1x_2-plane passing through the point $(x_1, x_2) = (4, 2)$. Specify a point on the x_1x_2-plane so that the trajectory passing through this point approaches 0 as $t \to \infty$.

   (d) Now, solve the non-homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ with initial conditions $x_1(0) = 2$ and $x_2(0) = 4$.

7. [15] Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Laplace Transform Table

$f(t)$	$F(s) = \mathbb{L}\{f(t)\}$	
1	$\frac{1}{s}$,	$s > 0$
e^{at}	$\frac{1}{s-a}$,	$s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}$,	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$,	$s > 0$