

The University of British Columbia  
Final Examination - December 7, 2005

**Mathematics 265**

All Sections

Closed book examination

Time: 2.5 hours

**Special Instructions:**

- Be sure that this examination booklet has 4 pages.
- No calculators or notes other than your one page formula sheet are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

**Rules governing examinations**

- Each candidate should be prepared to produce her/his library/AMS card upon request.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  - (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.
  - (b) Speaking or communicating with other candidates.
  - (c) Purposely exposing written papers to the view of other candidates.

1. [10] Consider the ODE  $y' = y(2 - y)$ .

(a) Draw a direction field for this ODE.

(b) Without solving the equation, determine  $\lim_{t \rightarrow \infty} y(t)$  if  $y(t)$  is a solution of this ODE with

i.  $y(0) = 0$ ,

ii.  $y(0) = 2$ ,

iii.  $y(0) = -1$ .

2. [10] Consider the ODE

$$(t - 1) \frac{dy}{dt} = \frac{2t}{t + 1} + 1,$$

Find the solution of this differential equation that satisfies  $y(0) = 0$ . Determine the largest interval in which your solution is valid.

3. [20] Consider the initial value problem

$$y'' + 4y = g(t) \quad y(0) = 3, \quad y'(0) = 1.$$

(a) Solve the corresponding homogeneous equation, and prove that the two solutions you have found are a fundamental set of solutions.

(b) For each of the following cases, what form would you guess for a particular solution  $Y(t)$ , if you were going to use the method of undetermined coefficients? **Do not solve the equation!**

i.  $g(t) = t^2 e^t$

ii.  $g(t) = e^t \sin t + t$

iii.  $g(t) = \cos(2t)$

(c) Solve the initial value problem when  $g(t) = 5 \cos(t)$ .

4. [15] Let  $f(t) = \sin(t)$  and  $g(t) = e^{-2t}$ .
- (a) Compute  $W(f, g)(t)$ .
  - (b) Are  $f$  and  $g$  linearly independent on  $(0, 2\pi)$ ? Justify your answer.
  - (c) Can one find an ODE  $y'' + p(t)y' + q(t)y = 0$  where  $p$  and  $q$  are continuous on  $(0, 2\pi)$  for which both  $f$  and  $g$  are solutions? Explain.
  - (d) Compute  $h(t) = f \star g$  where “ $\star$ ” denotes convolution.

5. [15] Solve the initial value problem

$$y'' + 3y' + 2y = (t - 2)u_2(t), \quad y(0) = y'(0) = 0.$$

What is  $\lim_{t \rightarrow \infty} y(t)$ ?

6. [15] Consider the system of first-order ODEs given by

$$\begin{aligned}x_1' &= -3x_1 + 4x_2 + 4e^{-t} \\x_2' &= -x_1 + 2x_2 + 3e^{-t}\end{aligned}$$

- (a) Write this system as a single matrix equation  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ .
- (b) Find the general solution of the homogeneous equation  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ .
- (c) Sketch the trajectory on the  $x_1x_2$ -plane passing through the point  $(x_1, x_2) = (4, 2)$ . Specify a point on the  $x_1x_2$ -plane so that the trajectory passing through this point approaches 0 as  $t \rightarrow \infty$ .
- (d) Now, solve the non-homogeneous system  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$  with initial conditions  $x_1(0) = 2$  and  $x_2(0) = 4$ .

7. [15] Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

## Laplace Transform Table

$f(t)$	$F(s) = \mathbb{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
$e^{at}$	$\frac{1}{s - a}, \quad s > a$
$\sin(at)$	$\frac{a}{s^2 + a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2 + a^2}, \quad s > 0$