Math 263 Final Exam (December 11, 2007)

<u>There are 6 Problems on both sides of the sheet.</u>

Problem 1: [15 points] Given the equations of two planes $3x+2y+z=4$ and $x-2y+z=1$,

(a) Prove that the two planes are perpendicular to each other.

(b) Find the parametric equation for their line of intersection.

(c) Find the symmetric equation for the line that goes through $(1,0,1)$ and is parallel to both planes.

Problem 2: [15 points] A space curve is specified by $\mathbf{r}(t) =< 2\cos t \sin t, 2\sin^2 t, t >$.

(a) Find the unit tangent vector at $t=0$.

(b) Find the curvature of the curve at $t=0$.

(c) Calculate the arc length from $t=0$ to $t=1$.

Problem 3: [15 points] Consider the function $f(x,y) = x^2 y$ defined in the domain D shown in Fig. 1. The domain D consists of a semicircle $x^2+y^2 \leq 1$ for $x \geq 0$ and a rectangle $[-1,0] \times [-1,1]$.

(a) Explain why $D_{\mathbf{u}} f(-1/2, 1/2) = -1$ is not possible for any unit vector $\mathbf{u}$.

(b) Find the absolute maximum and minimum values of $f(x,y)$ in D.

(c) Evaluate $\iint_D f(x,y)\, dA$.

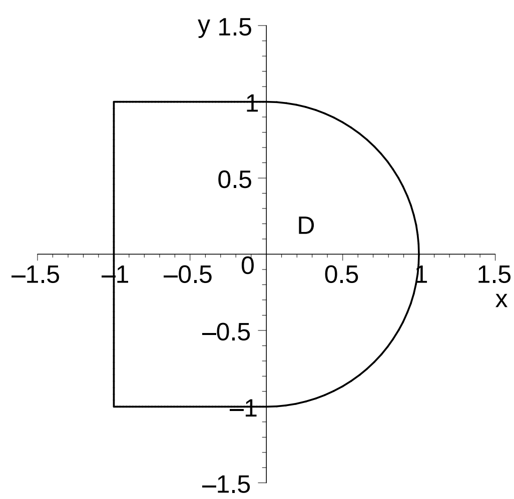

Figure 1: for Problem 3

Problem 4: [20 points] Let E be the solid region bounded by $z = \sqrt{x^2 + y^2}$ and $z = 2$ in the half-space $y \geq 0$.

(a) Set up the iterated integrals representing $\iiint_E (x^2 + y^2)z\, dV$ in the following orders of integration. Do not evaluate the integrals.

(i) $dz\, dy\, dx$

(ii) $dx\, dy\, dz$

(iii) $dz\, dr\, d\theta$ (cylindrical coordinates)

(b) Evaluate the triple integral $\iiint_E (x^2 + y^2)z\, dV$ by the method of your choice.

Problem 5: [20 points] Let $\mathbf{F}$ be the vector field $\mathbf{F} = \langle 2xy + ye^z \cos(x), x^2 + e^z \sin(x), e^z y \sin(x)\rangle$.

(a) Prove that $\mathbf{F}$ is conservative, and determine the potential f such that $\mathbf{F} = \nabla f$.

(b) Find $\operatorname{div} \mathbf{F}$. Can $\mathbf{F}$ be expressed as curl of another vector field?

(c) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the fundamental theorem for line integrals. Here C is part of the helix parameterised by the vector function $\mathbf{r}(t) = \langle t, \sin(t), \cos(t)\rangle$, and $0 \leq t \leq \pi/2$.

Problem 6: [15 points] Let $\mathbf{F} = \langle z, -x, y\rangle$, and let S be the closed surface formed by the paraboloid S_1: $z_1 = 12 - x^2 - y^2$ on top and the paraboloid S_2: $z_2 = 2x^2 + 2y^2$ at the bottom, with outward orientation.

(a) Find $\iint_{S_2} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.

(b) Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the Divergence theorem.