

**The University of British Columbia**  
Final Examination - December 17, 2013  
**Mathematics 257/316**  
All Sections

Closed book examination

Time: 2.5 hours

**Last Name** \_\_\_\_\_ **First** \_\_\_\_\_ **Signature** \_\_\_\_\_

**Student Number** \_\_\_\_\_

**Special Instructions:**

No books, notes, or calculators are allowed. A formula sheet is attached.

<b>Rules governing examinations</b>		
<ul style="list-style-type: none"><li>● Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.</li><li>● Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.</li><li>● No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.</li><li>● Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.</li><li>● Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:<ul style="list-style-type: none"><li>(a) speaking or communicating with other candidates, unless otherwise authorized;</li><li>(b) purposely exposing written papers to the view of other candidates or imaging devices;</li><li>(c) purposely viewing the written papers of other candidates;</li><li>(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,</li><li>(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).</li></ul></li><li>● Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.</li><li>● Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.</li><li>● Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).</li></ul>		

1		20
2		20
3		20
4		20
5		20
Total		100

1. Consider the differential equation

$$2x^2y'' - xy' + (1 + x^2)y = 0 \quad (1)$$

- (a) Classify the points  $0 \leq x < \infty$  as ordinary points, regular singular points, or irregular singular points.
- (b) Find two values of  $r$  such that there are solutions of the form  $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ .
- (c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

(Question 1 Continued)

(Question 1 Continued)

2. Consider a conducting metal bar of length  $\pi/2$  whose initial temperature is  $u(x, 0) = x$  and which loses heat to its surroundings. Assume that the left end of the bar is maintained at a zero temperature while the right end is insulated. The temperature distribution in the bar  $u(x, t)$  is determined by the following initial boundary value problem for the heat equation:

$$\begin{aligned}u_t &= u_{xx} - u, & 0 < x < \pi/2, & \quad t > 0 \\u(0, t) &= 0, & u_x(\pi/2, t) &= 0 \\u(x, 0) &= x\end{aligned}\tag{2}$$

- (a) Determine the solution to the boundary value problem (2) by separation of variables.

[14 marks]

- (b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to  $O(\Delta x^2, \Delta t)$  terms. Use the notation  $u_n^k \simeq u(x_n, t_k)$  to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition  $u_x(\pi/2, t) = 0$  with  $O(\Delta x^2)$  accuracy.

Hint: Taylor's expansion may prove useful:  $f(x + \Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + O(\Delta x^3)$ .

[6 marks]

[total 20 marks]

(Question 2 Continued)

(Question 2 Continued)

3. The motion of a damped string subject to an imposed load satisfies the following initial-boundary value problem:

$$u_{tt} + 2\gamma u_t = u_{xx} - 8 \sin x \cos x, \quad 0 < x < \pi, \quad t > 0 \quad (3)$$

$$u(0, t) = u(\pi, t) = 0 \quad (4)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin 3x.$$

- (a) Determine the static deflection  $w(x)$  of the string (i.e., the steady solution), which is determined by solving (3) with  $u_{tt} = u_t = 0$  and subject to the boundary conditions (4).

[5 marks]

- (b) Let  $u(x, t) = w(x) + v(x, t)$  and determine the corresponding boundary value problem for  $v(x, t)$ .

[5 marks]

- (c) Assuming that  $\gamma < 1$  use the method of separation of variables to solve for  $v(x, t)$  and therefore  $u(x, t)$ .

[10 marks]

[total 20 marks]

(Question 3 Continued)

(Question 3 Continued)

4. Consider the eigenvalue problem

$$\begin{aligned}x^2 y'' + xy' + \lambda y &= 0 \\ y'(1) &= 0 = y(e^{\pi/2})\end{aligned}$$

- (a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]
- (b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the semi-annular region:

$$\begin{aligned}u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 1 < r < e^{\pi/2}, \quad 0 < \theta < \pi \\ u(r, 0) &= 0 \quad \text{and} \quad u(r, \pi) = f(r) \\ \frac{\partial u(1, \theta)}{\partial r} &= 0 \quad \text{and} \quad u(e^{\pi/2}, \theta) = 0\end{aligned}$$

[12 marks]

[total 20 marks]

(Question 4 Continued)

(Question 4 Continued)

5. Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

$$\begin{aligned}u_t &= \alpha^2 u_{xx} + 1 - xe^{-t}, \quad 0 < x < 1, \quad t > 0 \\u_x(0, t) &= e^{-t}, \quad \text{and } u(1, t) = t \\u(x, 0) &= x.\end{aligned}$$

[20 marks]

(Question 5 Continued)

(Question 5 Continued)

(Additional Page)