Math 257/316 Final Exam, December 2011

Last Name: ____________________ First Name: ____________________

Student Number: ____________________ Signature: ____________________

Instructions. The exam lasts 2.5 hours. **No calculators or electronic devices of any kind are permitted**. A formula sheet is attached. There are **12 pages** in this test including this cover page, blank pages, and the formula sheet. Unless otherwise indicated, show all your work.

Rules governing formal examinations:

1. Each candidate must be prepared to produce, upon request, a UBC card for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
   - having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   - speaking or communicating with other candidates; and
   - purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Problem #	Value	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. [20 marks] For the ordinary differential equation

$$2x^2y'' + (3x + x^2)y' - y = 0,$$

find the first 3 terms of a non-zero series solution about $x = 0$ satisfying $\lim_{x\to 0+} y(x) = 0$.

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2. Consider the wave equation

$$u_{tt} = u_{xx},$$

with initial conditions

$$u(x,0) = f(x) = \begin{cases} 2\sin(2x) & 0 < x < \pi, \\ 0 & \text{otherwise} \end{cases} \qquad u_t(x,0) = 0.$$

(a) [10 marks] Suppose the domain is infinite, $-\infty < x < \infty$. Write down d'Alembert's solution, and sketch $u(x,0)$, $u(x,\pi/4)$, $u(x,\pi/2)$, and $u(x,3\pi/4)$.

(b) [10 marks] Suppose instead the domain is the interval $[0,\pi]$, with zero (Dirichlet) boundary conditions:

$$u(0,t) = 0 = u(\pi,t),$$

and find the solution by separation of variables.

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3. (a) [12 marks] Find both the Fourier sine series and the Fourier cosine series of the function $f(x) = x$ on the interval $[0, 1]$.

   (b) [8 marks] According to the Fourier Convergence Theorem, for which values of x in the interval should each of these series converge to x? Verify your conclusions for the Fourier sine series at $x = 0$ and $x = 1$.

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4. Consider the following problem for the heat equation with a time-dependent source term, and mixed boundary conditions:

$$u_t = u_{xx} + t, \quad 0 < x < 1, \ t > 0,$$

$$u_x(0,t) = 0, \quad u(1,t) = 0, \qquad u(x,0) = 1.$$

(a) [7 marks] Briefly describe how you would use the method of finite differences to find an approximate solution to this problem. Use the notation $u_n^k \approx u(x_n, t_k)$ to denote the values of u on the finite difference mesh, and include how you propose to incorporate the boundary and initial conditions. *In case it is useful, the Taylor expansion formula is* $f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)(\Delta x)^2 + O((\Delta x)^3)$.

(b) [13 marks] Find the solution to this problem using the method of eigenfunction expansion.

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5. Consider the following problem involving Laplace's equation in an annular region:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2,\ 0 < \theta < \pi/2,$$

$$u(1,\theta) = 0, \quad u(2,\theta) = 0, \qquad u(r,0) = 0, \quad u(r,\pi/2) = f(r).$$

(a) [10 marks] Use the method of separation of variables to solve the problem when $f(r) = \sin\left(\frac{2\pi}{\ln(2)}\ln(r)\right)$.

(b) [10 marks] Find the solution for a general function $f(r)$.

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Math 257-316 PDE   Formula sheet - final exam

Trigonometric and Hyperbolic Function identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ $\sin^2 t + \cos^2 t = 1$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha.$ $\sin^2 t = \frac{1}{2}(1-\cos(2t))$

$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \sinh\beta\cosh\alpha$ $\cosh^2 t - \sinh^2 t = 1$

$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\beta\sinh\alpha.$ $\sinh^2 t = \frac{1}{2}(\cosh(2t)-1)$

Basic linear ODE's with real coefficients

	constant coefficients	**Euler eq**
ODE	$ay''+by'+cy=0$	$ax^2y''+bxy'+cy=0$
indicial eq.	$ar^2+br+c=0$	$ar(r-1)+br+c=0$
$r_1 \neq r_2$ real	$y = Ae^{r_1x}+Be^{r_2x}$	$y = Ax^{r_1}+Bx^{r_2}$
$r_1 = r_2 = r$	$y = Ae^{rx}+Bxe^{rx}$	$y = Ax^r + Bx^r\ln\lvert x\rvert$
$r = \lambda \pm i\mu$	$e^{\lambda x}[A\cos(\mu x)+B\sin(\mu x)]$	$x^{\lambda}[A\cos(\mu\ln\lvert x\rvert)+B\sin(\mu\ln\lvert x\rvert)]$

Series solutions for $y''+p(x)y'+q(x)y=0$ $(\star)$ around $x=x_0$.

Ordinary point x_0: Two linearly independent solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

Regular singular point x_0: Rearrange $(\star)$ as:
$(x-x_0)^2y'' + [(x-x_0)p(x)](x-x_0)y' + [(x-x_0)^2q(x)]y = 0$
If $r_1 > r_2$ are roots of the indicial equation: $r(r-1)+br+c=0$ where $b = \lim_{x\to x_0}(x-x_0)p(x)$ and $c = \lim_{x\to x_0}(x-x_0)^2q(x)$ then a solution of $(\star)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+r_1} \quad \text{where } a_0 = 1.$$

The second linerly independent solution y_2 is of the form:
Case 1: If $r_1 - r_2$ is neither 0 nor a positive integer:

$$y_2(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2} \text{ where } b_0 = 1.$$

Case 2: If $r_1 - r_2 = 0$:

$$y_2(x) = y_1(x)\ln(x-x_0) + \sum_{n=1}^{\infty} b_n(x-x_0)^{n+r_2} \text{ for some } b_1, b_2...$$

Case 3: If $r_1 - r_2$ is a positive integer:

$$y_2(x) = ay_1(x)\ln(x-x_0) + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2} \text{ where } b_0 = 1.$$

Fourier, sine and cosine series

Let $f(x)$ be defined in $[-L,L]$then its Fourier series $Ff(x)$ is a $2L$-periodic function on $\mathbf{R}$: $Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})\right\}$
where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx$

Theorem (Pointwise convergence) If $f(x)$ and $f'(x)$ are piecewise continuous, then $Ff(x)$ converges for every x to $\frac{1}{2}[f(x-)+f(x+)]$.

Parseval's indentity

$$\frac{1}{L}\int_{-L}^{L} |f(x)|^2dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right).$$

For $f(x)$ defined in $[0,L]$, its cosine and sine series are

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx,$$

$$Sf(x) = \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx.$$

D'Alembert's solution to the wave equation

PDE: $u_{tt} = c^2u_{xx}$, $-\infty < x < \infty$, $t > 0$ **IC**: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.
SOLUTION: $u(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$

Sturm-Liouville Eigenvalue Problems

ODE: $[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad a < x < b.$
BC: $\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0.$
Hypothesis: p, p', q, r continuous on $[a,b]$. $p(x) > 0$ and $r(x) > 0$ for $x \in [a,b]$. $\alpha_1^2 + \alpha_2^2 > 0$. $\beta_1^2 + \beta_2^2 > 0$.
Properties (1) The differential operator $Ly = [p(x)y']' - q(x)y$ is symmetric in the sense that $(f, Lg) = (Lf, g)$ for all f, g satisfying the BC, where $(f,g) = \int_a^b f(x)g(x)\,dx$. (2) All eigenvalues are real and can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$, and each eigenvalue admits a unique (up to a scalar factor) eigenfunction ϕ_n.
(3) **Orthogonality**: $(\phi_m, r\phi_n) = \int_a^b \phi_m(x)\phi_n(x)r(x)\,dx = 0$ if $\lambda_m \neq \lambda_n$.
(4) **Expansion**: If $f(x) : [a,b] \to \mathbf{R}$ is square integrable, then

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x),\ a < x < b\ ,\ c_n = \frac{\int_a^b f(x)\phi_n(x)r(x)\,dx}{\int_a^b \phi_n^2(x)r(x)\,dx},\ n = 1,2,\ldots$$