The University of British Columbia
Final Examination - December 9, 2005

Mathematics 257/316

Sections 101, 102, 103
Instructors: Dr. Brydges, Dr Sjerve, Dr. Tsai

Closed book examination — Time: 2.5 hours

Name ______________________ **Signature** ______________________

Student Number ______________

Special Instructions:
- Be sure that this examination has 10 pages. Write your name on top of each page.
- A formula sheet is provided. No calculators or notes are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate should be prepared to produce her/his library/AMS card upon request.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates.
- Smoking is not permitted during examinations.

1		20
2		15
3		15
4		15
5		20
6		15
Total		100

[20] 1. Consider the equation

$$2x^2y'' - xy' + (1+x)y = 0$$

(a) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(b) Find the recurrence relation for a_n in terms of a_{n-1} for both values of r. (You may do both at the same time by not substituting values for r in the recurrence relation).

(c) For the larger of the two values of r and for $a_0 = 1$ find a_1, a_2, a_3.

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[15] 2. Solve the heat equation

$$u_t(x,t) = u_{xx}(x,t)$$

for $0 \leq x \leq 1$ and $t \geq 0$ with non-homogeneous boundary conditions

$$u(0,t) = 1, \quad u(1,t) = 0, \quad \forall t > 0$$

and initial condition

$$u(x,0) = 1 + x.$$

[15] 3. Consider the wave equation $u_{tt} = u_{xx}$ for $x \in \mathbf{R}$, $t > 0$, with the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, where

$$f(x) = \begin{cases} 0 \text{ if } x \leq 0, \\ x \text{ if } 0 \leq x \leq 1, \\ 1 \text{ if } x \geq 1 \end{cases} \qquad g(x) = \begin{cases} 0 \text{ if } x < 0, \\ 1 \text{ if } 0 \leq x \leq 1, \\ 0 \text{ if } x > 1. \end{cases}$$

(a) Solve the solution $u(x,t)$ using d'Alembert's formula. Simplify it as much as possible. Hint: draw the graph of $f'(x)$, where it exists.

(b) Plot $u(x,0)$, $u(x,1)$ and $u(x,2)$ for $|x| \leq 4$.

[15] 4. Solve the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y) = 0$$

for $0 \le x \le \pi$ and $0 \le y \le 1$ with boundary conditions

$$u(0,y) = \sin(\pi y), \quad u(\pi,y) = 0, \quad u(x,0) = \sin^3 x, \quad u(x,1) = 0.$$

Hint: $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$.

[20] 5. Consider the boundary value problem

$$x^2y'' + xy' + \lambda y = 0, \quad y'(1) = 0, \quad y'(e) = 0.$$

(a) Find all values of λ such that this problem has a non-zero solution $y(x)$. For each value of λ you find give a non-zero solution $y(x)$.

(b) Let $y_j(x)$, $y_k(x)$ be non-zero solutions corresponding to different values of λ in part (a). Is there an orthogonality relation satisfied by $y_j(x)$ and $y_k(x)$, and if so what is it? (This can be answered without answering part (a)).

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[15] 6. Consider the problem

$$\Delta u + u = 0 \text{ on the disc } 0 < r < 1,\ 0 \le \theta \le 2\pi,\ u(1,\theta) = f(\theta) \text{ on the boundary.}$$

(a) Use the separation of variables method to find all solutions of the form $u(r,\theta) = R(r)\Theta(\theta)$, which are finite at the origin. Show your work.

(b) Find the solution if $f(\theta) = \sin 6\theta$.

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