Math 256. Final

Name:

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $y' - y^2 e^{-x} = y^2$ with $y(0) = 1$ has the solution,

$$(a)\quad (e^x + x + C)^{-1} \qquad (b)\quad (e^{-x} - x + C)^{-1} \qquad (c)\quad (e^{-x} - x)^{-1}$$

$$(d)\quad (e^{-x} + x)^{-1} \qquad (e)\quad \textit{None of the above},$$

where C is an arbitrary constant.

2. The system

$$\mathbf{y}' = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{y}$$

has the general solution,

$$(a)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{t} \qquad (b)\quad \mathbf{u}_1 e^{7t} + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{-2t} \qquad (c)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{7t}$$

$$(d)\quad \mathbf{u}_1 e^{-t} + \mathbf{u}_2 e^{4t} + +\mathbf{u}_3 e^{2t} \qquad (e)\quad \textit{None of the above},$$

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of

$$\bar{y}(s) = \frac{4}{s(s^2+4)}$$

$$(a)\quad y(t) = t - \cos 2t, \qquad (b)\quad y(t) = t + \sin 2t, \qquad (c)\quad y(t) = 1 - \cos 2t,$$

$$(d)\quad y(t) = 1 - \sin 2t, \qquad (e)\quad \textit{None of the above.}$$

4. Which of the following is a solution to the PDE $u_{tt} = c^2 u_{xx}$:

$$(a)\quad u = \cos cx\ \sin t \qquad (b)\quad u = \cos x\ \sin t \qquad (c)\quad u = \cos 3x\ \sin 3ct$$

$$(d)\quad u = e^x\ \sin ct \qquad (e)\quad \textit{None of the above.}$$

5. Laplace transforms are

(a) *Cool if you like that sort of thing*

(b) *difficult to invert using formal definitions*

(c) *a type of integral*

(d) *a method of turning ODEs into algebraic equations*

(e) *All of the above.*

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* A particle in a fusion reactor satisfies the equations of motion,

$$x' + x + 2y = 0, \quad y' + y - 2x = 0, \quad z' + z = x^2 + y^2.$$

Find the path taken by the particle if it starts at the point $(x(0), y(0), z(0)) = (0, 1, 0)$. Where does the particle eventually end up?

2. *(12 points)* Write the ODEs

$$x'' = x + 4y, \quad y'' = 2x + 8y,$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. Find the solution if the initial values are $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 9$.

3. *(12 points)* From the definition of the Laplace transform, prove that $\mathcal{L}\{y''\} = s^2\bar{y}(s) - sy(0) - y'(0)$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s)$. Find

$$\mathcal{L}^{-1}\left\{\frac{s+4}{(s^2+4s+13)}\right\}$$

Using Laplace transforms, solve the ODE

$$\ddot{y} + 4\dot{y} + 13y = t^2\delta(t-2),$$

with $y(0) = 1$ and $\dot{y}(0) = 0$, where $\delta(t)$ is the delta-function.

4. *(16 points)* Solve

$$u_t = u_{xx}, \quad u(0,t) = 0, \quad u(x,0) = \sin \pi x,$$

with

$$(a) \quad u(2,t) = 0 \qquad \& \qquad (b) \quad u(2,t) = 2.$$

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Helpful trig identities:

$$\begin{gathered}
\sin 0 = \sin\pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2), \\
\cos 0 = -\cos\pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0, \\
\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1, \\
\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B, \\
\cos(2A) = \cos^2 A - \sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,
\end{gathered}$$

Useful Laplace Transforms:

$$f(t) \quad \rightarrow \quad \bar{f}(s)$$

$$1 \quad \rightarrow \quad 1/s$$

$$t^n, \quad n = 0, 1, 2, ... \quad \rightarrow \quad n!/s^{n+1}$$

$$e^{at} \quad \rightarrow \quad 1/(s-a)$$

$$\sin at \quad \rightarrow \quad a/(s^2+a^2)$$

$$\cos at \quad \rightarrow \quad s/(s^2+a^2)$$

$$t\sin at \quad \rightarrow \quad 2as/(s^2+a^2)^2$$

$$t\cos at \quad \rightarrow \quad (s^2-a^2)/(s^2+a^2)^2$$

$$y'(t) \quad \rightarrow \quad s\bar{y}(s) - y(0)$$

$$y''(t) \quad \rightarrow \quad s^2\bar{y}(s) - y'(0) - sy(0)$$

$$e^{at}f(t) \quad \rightarrow \quad \bar{f}(s-a)$$

$$f(t-a)H(t-a) \quad \rightarrow \quad e^{-as}\bar{f}(s)$$