Final Examination — December 9, 2016   Duration: 2.5 hours
*This test has **8 questions** on **12 pages**, for a total of 80 points.*

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- Read all the questions carefully before starting to work. Unless otherwise indicated, give complete arguments and explanations for all your calculations as answers without justification will not be marked.

- Continue on the back of the *previous page* if you run out of space, *with clear indication on the original page* that your solution is continued elsewhere.

- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: _____   Last Name: _____

Student-No: _____   Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	Total
Points:	14	15	10	6	7	14	8	6	80
Score:									

5 marks   1.   (a) Suppose $z = f(x, y)$ describes the surface of a mountain. The values of some derivatives are known at certain points:

point	f_x	f_y	f_{xx}	f_{yy}	f_{xy}
A	0	1	4	-3	-2
B	0	0	4	3	-4
C	10	2	-5	3	2
E	0	0	-3	-3	2
F	0	0	5	5	5
G	0	0	3	4	3

For each situation, identify a suitable point **and *briefly* explain:** e.g., "X, because it is a local minimum." **Justify your answers with calculations.**

    i. Mika wants to hike to point with view in every direction (where the mountain does not obscure her view). Where should she go?

Answer:

    ii. Xiaofei is a biologist who wants to re-introduce a species of frog into a wet environment. Where is water mostly likely to form a pool?

Answer:

    iii. Donnie wants to have picnic at a place where the ground is flat and dry. He thinks a peak would be too windy. Where should he go?

Answer:

    iv. Akshat also wants to have a picnic where the ground is flat, but he likes surprises. Where should he go?

Answer:

2 marks   (b) Suppose $f(x, y)$ is some function, unrelated to previous problem. At a point (x, y), what is the maximal value of the directional derivative $D_{\vec{u}}f$?

Answer:

7 marks   (c) Suppose $f(x, y)$ is some function. Define $g(x, y) = |\vec{\nabla}f|^2 = (f_x(x, y))^2 + (f_y(x, y))^2$.

        i. What is $\vec{\nabla}g$?

        ii. Consider a mountain where the derivatives are $f_x(x, y) = \exp(-x^2)$ and $f_y(x, y) = \exp(-y^4 + 2y^2)$. Find the critical points of $g(x, y)$ (no need to classify).

Answer:

        iii. What is the geometric interpretation (in terms of f) of the point (x, y) where $g(x, y)$ is maximal?

2 marks 2. (a) To find the surface area of the surface $z = f(x,y)$ above the region D, we integrate $\iint_D F(x,y)\, dA$. What is $F(x,y)$?
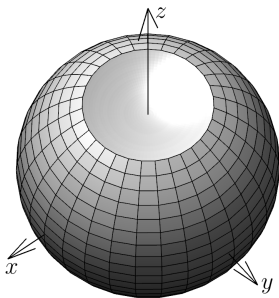
Answer:

(b) Consider a "Death Star", a ball of radius 2 centred at the origin with another ball of radius 2 centred at $(0,0,2\sqrt{3})$ cut out of it. The diagram shows the slice where $y = 0$.

4 marks   i. The Rebels want to paint *part* of the surface of Death Star hot pink; specifically, the concave part (indicated with a thick line in the diagram). To help them determine how much paint is needed, carefully fill in the missing parts of this integral:
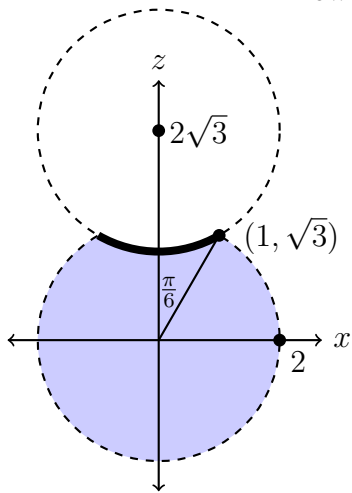
$$\text{surface area} = \int_{\square}^{\square} \int_{\square}^{\square} \boxed{}\, dr\, d\theta.$$
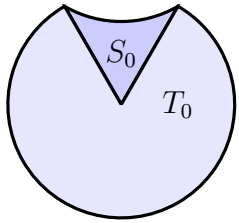

1 mark   ii. What is the total surface area of the Death Star?
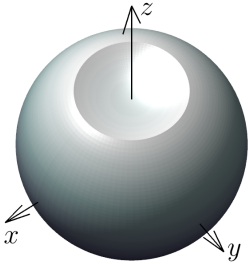
Answer:

8 marks   (c) Suppose the solid interior of the Death Star has density T_0, except in a cone connecting the origin to the concave region where the density is S_0. Complete this expression for the total mass M:

$$M = \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} T_0 \boxed{} \, d\rho \, d\phi \, d\theta$$

$$+ \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} S_0 \boxed{} \, dz \, dr \, d\theta$$

2 marks 3. (a) Give the formula for the linear approximate of $h(x,y)$ near the point (x_0, y_0).

$$h(x,y) \approx$$

4 marks (b) The equation $z^5 - 2z^3 - x^2 y - 7y = -33$ determines a function $z = h(x,y)$ implicitly where $h(3,2) = 1$. Find the linear approximation of h near $(x_0, y_0) = (3,2)$.

Answer:

4 marks (c) The kinetic energy K of an object rotating with angular velocity $\omega = 3$ radian per second and moment of inertia $I = 10\,\text{Kg m}^2$ is given by $K = \frac{1}{2}I\omega^2$. However, the moment of inertia has an error of up to $\pm 0.1\,\text{Kg m}^2$ and the angular velocity has an error of up to ± 0.02 radian per second. Use differentials to estimate the maximal possible error in the computed kinetic energy K.

Answer:

6 marks   4. Match the following functions with their contour plot.

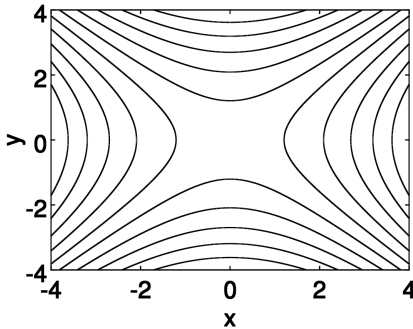$$\underline{\hspace{1cm}}\ f(x,y) = xe^{-x^2-y^2}\,, \qquad \underline{\hspace{1cm}}\ f(x,y) = \sin(x) - \cos(y)\,, \qquad \underline{\hspace{1cm}}\ f(x,y) = 3x^2 - y\,,$$

$$\underline{\hspace{1cm}}\ f(x,y) = \sqrt{12 - 4x^2 - y^2}\,, \qquad \underline{\hspace{1cm}}\ f(x,y) = xy\,. \qquad \underline{\hspace{1cm}}\ f(x,y) = x^2 - y^2\,.$$
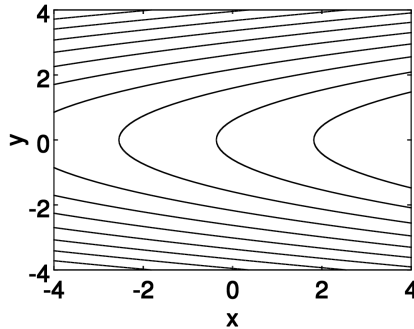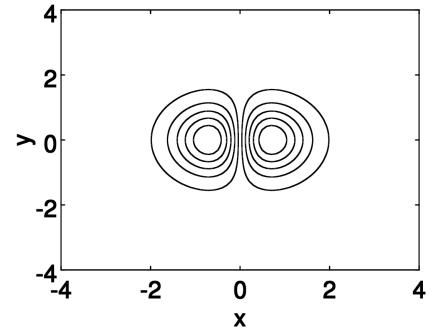
(A)   (B)   (C)
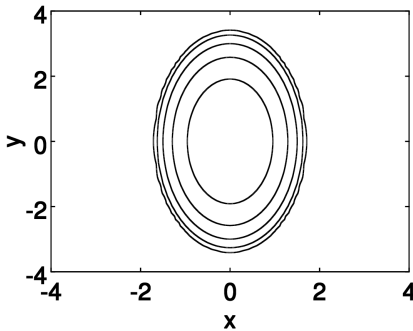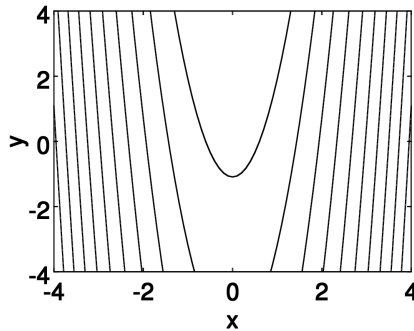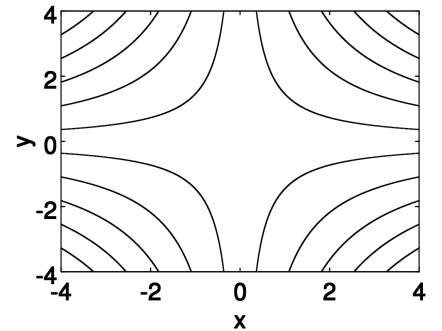
(D)   (E)   (F)
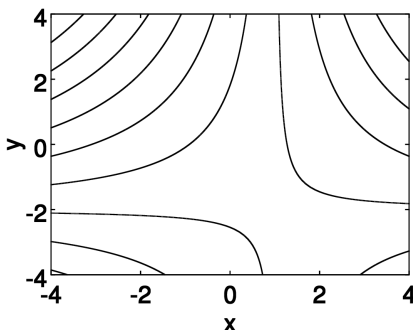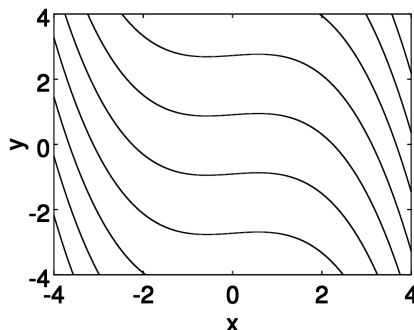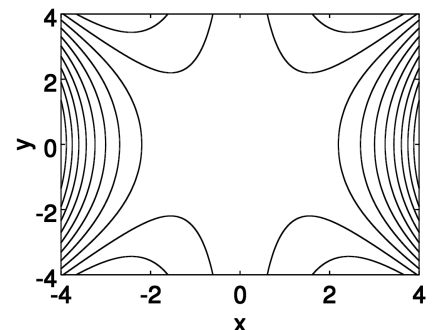
(G)   (H)   (I)

(J)   (K)   (L)

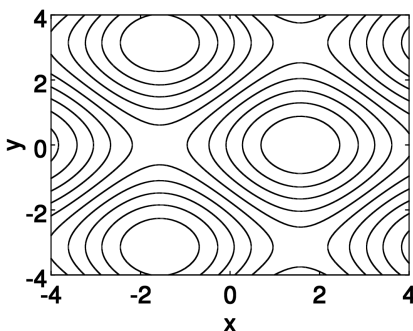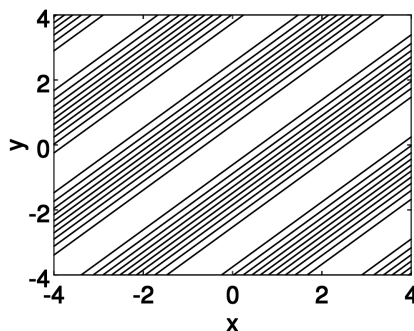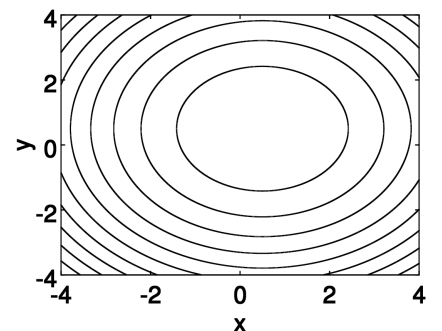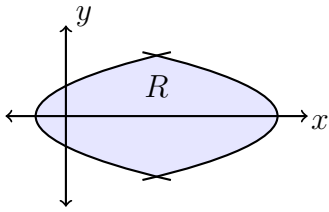4 marks   5.   (a) Let R be the region bounded on the left by $x = y^2 - 1$ and on the right by $x = -y^2 + 7$. Evaluate the integral $\iint_R 3 \, dA$.

Answer:

1 mark   (b) What is the area of R?

Answer:

2 marks   (c) Evaluate the integral

$$\int_{-2}^{2} \int_{y^2-1}^{-y^2+7} y \cos\left(y^6\right) \cos(x) \, dx \, dy =$$

Answer:

8 marks　6. (a) Let

$$f(x, y, z) = e^{\left((x^2+y^2+z^2)^{3/2}\right)}.$$

Rewrite

$$\int_0^3 \int_{-\sqrt{(9-y^2)}}^{\sqrt{(9-y^2)}} \int_{\sqrt{(x^2+y^2)}}^{\sqrt{(18-x^2-y^2)}} f(x, y, z)\, dz\, dx\, dy$$

using spherical coordinates. It *is not necessary* to evaluate.

6 marks   (b) Let

$$\int_0^4 \int_{-2}^{2-z} \int_{-\sqrt{4-y^2}}^0 f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, .$$

Change the order of integration to be $\mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$.

8 marks   7. Calculate the volume of the region E enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

8. Consider the planes $z = 4x - 2y - 3$ and $z = 8x - 4y - 12$.

2 marks   (a) Determine the cosine of the acute angles between the two planes.

4 marks   (b) Determine the equation of the line of intersection of the two planes. Give your answer in parametric form. At what point (x, y, z) does that line intersect with the plane $x = 1$?

Answer: