Department of Mathematics
University of British Columbia
MATH 227 (Section 201) Final Exam
April 26, 2011, 8:30 AM - 11:00

Family Name: ______________________________ Initials: ______

I.D. Number: ________________ Signature: ______________________________

CALCULATORS, NOTES OR BOOKS ARE NOT PERMITTED.

JUSTIFY ALL OF YOUR ANSWERS (except as otherwise specified).

THERE ARE 8 PROBLEMS ON THIS EXAM.

Question	**Mark**	**Out of**
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
Total		80

1.

Let $\mathbf{x} : [1, 2] \to \mathbb{R}^3$ be defined by:

$$\mathbf{x}(t) = \left(\frac{t^2}{2\sqrt{2}}, \frac{t^2}{2\sqrt{2}}, \frac{t^3}{3} \right).$$

(a) Compute the arc length of the path $\mathbf{x}$ from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ for all $1 \leq t \leq 2$.

(b) Find an explicit reparametrization of $\mathbf{x}$ by arc length.

2.

Which of the following subsets of $\mathbb{R}^3$ is simply connected? (note that such a set need not be open).

Just answer Yes or No (justification is *not* required).

(a) The complement of a line

(b) The complement of a half-line

(c) The complement of a circle

(d) The complement of the unit ball

(e) $\{(x, y, z) \in \mathbb{R}^3 : \ 2 < x^2 + y^2 + z^2 < 3\}$

(f) The unit sphere

(g) The punctured unit sphere (i.e., the set consisting of all points in the unit sphere except for one point)

(h) The doubly punctured unit sphere (i.e., the set consisting of all points in the unit sphere except for two distinct points)

(i) The torus

(j) The solid torus (i.e., the bounded solid region that is bounded by the torus)

3.

Let $C_r(\mathbf{p})$ denote the circle of radius r centered at $\mathbf{p} \in \mathbb{R}^2$, oriented counterclockwise.

Let $\mathbf{F} = (M, N)$ be a C^1 vector field on $\mathbb{R}^2$ such that $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ at all points *except* possibly $(0,0), (0,2)$ and $(0,-2)$. Let

$$\begin{aligned}
\alpha &= \int_{C_3((0,2))} \mathbf{F} \cdot d\mathbf{s} \\
\beta &= \int_{C_3((0,-2))} \mathbf{F} \cdot d\mathbf{s} \\
\gamma &= \int_{C_5((0,0))} \mathbf{F} \cdot d\mathbf{s} \\
\delta &= \int_{C_1((0,0))} \mathbf{F} \cdot d\mathbf{s}
\end{aligned}$$

Find δ in terms of α, β and γ.

4.

Let C_1 be the oriented straight line from $(0,0,0)$ to $(0,1,0)$. Let C_2 be the oriented straight line from $(0,1,0)$ to $(1,0,1)$. Let C be the curve consisting of C_1 followed by C_2.

Let $\mathbf{F}$ be the vector field:

$$\mathbf{F}(x,y,z) = (-xy + x^3 + z^2,\ e^{y^5 \cos(y)} - z - x,\ yz + e^z).$$

Find $\int_C \mathbf{F} \cdot d\mathbf{s}$.

5.

Let B be the unit ball and S be the unit sphere in $\mathbb{R}^3$. Let $\mathbf{F}$ be a C^1 vector field on a neighbourhood of B. Assume that

i. $curl(\mathbf{F}) = \mathbf{0}$

ii. $div(\mathbf{F}) = \mathbf{0}$ – *and* –

iii. On S, $\mathbf{F}$ is orthogonal to the radial vector field (x, y, z).

Show that $\mathbf{F} = \mathbf{0}$ on B.

(You may want to use the fact that if the integral of a nonnegative continuous function is zero, then the function itself is identically zero).

6.

For $i = 1, 2, 3$, let

$$\phi_i = \sum_{j=1}^{3} f_{ij} dx_j$$

be a differential 1-form on $\mathbb{R}^3$. Show that

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \det\left(\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}\right) dx_1 \wedge dx_2 \wedge dx_3$$

7.

Let S denote the torus in $\mathbb{R}^3$ defined by:

$$(\sqrt{x^2+y^2}-2)^2+z^2=1.$$

Find an orientation form on S, i.e., a differential 2-form on S that is nonzero on every pair of linearly independent tangent vectors at every point of S. Give your answer in the form

$$f_1\,dx\wedge dy+f_2\,dy\wedge dz+f_3\,dz\wedge dx,$$

with explicit expressions for f_1, f_2, and f_3 as functions of x, y and z.

8.

Let $\mathbf{F}(x) = e^{x^6}(x - x^3)$, considered as a vector field on $\mathbb{R}^1 = \mathbb{R}$.

(a) Find the stationary points of $\mathbf{F}$ (recall that a *stationary point* of a vector field is a point $\mathbf{p}$ such that the flow line through $\mathbf{p}$ consists of the single point $\mathbf{p}$ itself).

(b) Sketch $\mathbf{F}$.

(c) Let $\mathbf{x}(t)$ be any flow line for $\mathbf{F}$. Prove that $\lim_{t\to+\infty} \mathbf{x}(t)$ is a stationary point of $\mathbf{F}$ (you do not need to prove that the limit exists).

(d) For a point $\mathbf{q} \in \mathbb{R}$, let $\mathbf{x}^{\mathbf{q}}(t)$ denote the flow line such that $\mathbf{x}^{\mathbf{q}}(0) = \mathbf{q}$. For a stationary point $\mathbf{p}$ of $\mathbf{F}$, define the *basin of attraction*:

$$B_{\mathbf{p}} = \{\mathbf{q} \in \mathbb{R} : \lim_{t\to+\infty} \mathbf{x}^{\mathbf{q}}(t) = \mathbf{p}\}.$$

Explicitly describe the basin of attraction for each stationary point of $\mathbf{F}$.