This examination has 9 questions on 3 pages.

The University of British Columbia
Final Examinations—December 2010
Mathematics 226
Advanced Calculus I (*Professor Loewen*)

Closed book examination. Time: 2.5 hours

Notes, books, and calculators are not allowed.
Write your answers in the booklet provided. Start each solution on a separate page.

SHOW ALL YOUR WORK!

[10] **1.** The plane P passes through the points $A(1,1,0)$, $B(0,1,1)$, and $C(1,3,1)$.

(a) Find an equation for P.

(b) Show how to split the given vector $\mathbf{F} = (-4, 5, -7)$ as

$$\mathbf{F} = \mathbf{u} + \mathbf{w},$$

where $\mathbf{u}$ is normal to the plane P and $\mathbf{w}$ is perpendicular to $\mathbf{u}$.

[8] **2.** Riverboat Slim earns his living as a calculus shark on the paddlewheelers of the Mississippi. An undercover cop once learned Slim's trick for finding the tangent plane for any surface of the form

$$Ax^2 + By^2 + Cz^2 = K. \tag{$*$}$$

"If (x_0, y_0, z_0) is on that surface and I need the tangent plane right there", Slim confided, "I just rewrite $(*)$ but change x^2 to x_0x, change y^2 to y_0y, and so on." The cop got the hint:

$$Ax_0x + By_0y + Cz_0z = K. \tag{$**$}$$

(a) Justify Slim's method.

(b) Generalize Slim's method to produce the tangent hyperplane at $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ for the $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$ defined by

$$c_1x_1^p + c_2x_2^p + \cdots + c_nx_n^p = K. \tag{$\dagger$}$$

(Here $K \in \mathbb{R}$, $\mathbf{c} \in \mathbb{R}^n$, and $p > 1$ are given constants; point $\mathbf{a}$ lies on the surface.) Your goal is an equation named $(\ddagger)$ that is related to $(\dagger)$ in the same way that Slim's equation $(**)$ is related to $(*)$.

[12] **3.** (a) Use Lagrange Multipliers to minimize $f(x, y, z) = x^2 + y^2$ subject to the simultaneous constraints

$$x + 2y + 3z = 6 \qquad \text{and} \qquad x - y - 3z = -1.$$

[A correct solution that does not use Lagrange Multipliers will earn some partial credit.]

(b) Discuss the problem produced by changing "Minimize" to "Maximize" in part (a).

[10] **4.** Evaluate $I = \displaystyle\int_0^1 \int_y^1 x^2 \sin(\pi xy)\, dx\, dy.$

[15] **5.** Near Saddleback Pass, the elevation z is related to the map coordinates (x, y) by

$$z = f(x, y) \stackrel{\text{def}}{=} z_0 - \frac{x^2}{4} + \frac{y^2}{8}.$$

We have volunteered to build a trail from the point A, where $x = 1$ and $y = 0$, to the mountain pass at B, where $x = 0$ and $y = 0$. Our instructions specify that

(i) the trail must start out in the direction of increasing y, and

(ii) hikers on the trail must not face a slope larger than $1/4$.

(a) Sketch a contour map showing level curves of f near the mountain pass. Make a sketch that is large enough and neat enough to support the additions described below.

(b) If we choose a slope of exactly $1/4$ to meet rule (ii), in what direction does the trail leave point A? Show this direction on your map from (a).

(c) If an extreme runner sprints uphill along our trail, maintaining a speed of $\|\mathbf{v}\| = 2$, what will be his/her three-dimensional velocity vector $\mathbf{v}$ at the point A? [Assume all coordinates are measured in meters and the given speed is in meters per second.]

(d) Near the pass at B, the ground is almost level, so *rule (ii) is satisfied for motion in any direction at all.* Find all points where this is true, and show them on your map from (a).

[12] **6.** (a) Write the precise logical definition of the statement below:

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L. \qquad (*)$$

(b) If the following limit exists, evaluate it and prove in detail that the definition in part (a) holds. Otherwise, prove in detail that existence fails.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 + x^2 y - y^2 x + 2y^2}{x^2 + y^2}.$$

(c) Apply the instructions from part (b) to

$$\lim_{(x,y)\to(0,1)} \frac{x^2y^2 - 2x^2 y + x^2}{\left(x^2 + y^2 - 2y + 1\right)^2}.$$

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[13] **7.** All parts of this question concern the function $f(x,y) = y^2 - y\sin\left(\frac{\pi}{2}x\right)$ and the related set $C \stackrel{\text{def}}{=} \{(x,y) : f(x,y) = 0\}$.

(a) Make a good sketch, with labels, of the part of C where $-1 \le x \le 3$.

(b) Let $Q(x,y)$ denote the best quadratic approximation for $f(x,y)$ near the point $(1,1)$.

(i) Write a formula for $Q(x,y)$.

(ii) The equation $Q(x,y) = 0$ provides an ellipse that approximates the shape of C near $(1,1)$. Write the equation for this ellipse in the form

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

and sketch it on the figure produced in part (a).

(c) Now, instead, let $Q(x,y)$ denote the best quadratic approximation for $f(x,y)$ near $(2,0)$.

(i) Write a formula for $Q(x,y)$.

(ii) Discuss the approximation for C near $(2,0)$ provided by the equation $Q(x,y) = 0$.

[16] **8.** All parts of this question refer to $J = \displaystyle\int_{-4}^{4}\int_{9}^{25-x^2}\int_{-\sqrt{25-x^2-z}}^{\sqrt{25-x^2-z}} f(x,y,z)\,dy\,dz\,dx.$

(a) Reiterate J in all possible orders.

(b) Evaluate J when $f(x,y,z) = xze^y$.

(c) Evaluate J when $f(x,y,z) = \cos(\pi[x^2+y^2])$.

[4] **9.** Prove that for any 26 real numbers $x_1, x_2, \ldots, x_{26}$,

$$(x_1 + x_2 + \ldots + x_{26})^2 \le 26\left(x_1^2 + x_2^2 + \ldots + x_{26}^2\right).$$

(Any logically correct and clearly justified method earns full marks. The question's point value suggests that an easy method exists.)

[100] **Total Marks Available**