This examination has 8 questions on 2 pages.

The University of British Columbia
Final Examinations—December 2003

Mathematics 226
Advanced Calculus I (Professor Loewen)

Closed book examination. Time: $2\frac{1}{2}$ hours

Notes, books, and calculators are not allowed.
Write your answers in the booklet provided. Start each solution on a separate page.

SHOW ALL YOUR WORK!

[10] **1.** A nonzero vector $\mathbf{c}$ in $\mathbb{R}^3$ is given, along with constants a, k obeying $|k| < a|\mathbf{c}|$. Show that the plane $\mathbf{c} \bullet \mathbf{x} = k$ and the sphere $|\mathbf{x}| = a$ intersect in a circle. Find the centre and radius of the circle in terms of $a, k, \mathbf{c}$.

[10] **2.** Show that the curve of intersection

$$C: \qquad x^2 - y^2 + z^2 = 1, \quad xy + xz = 2$$

meets the following surface tangentially at the point $(1, 1, 1)$:

$$S: \qquad xyz - x^2 - 6y = -6.$$

[12] **3.** Functions $f, g \colon \mathbb{R}^4 \to \mathbb{R}$ are of class C^1, and obey

$$f(\mathbf{0}) = 0 = g(\mathbf{0}), \qquad \nabla f(\mathbf{0}) = (2, 3, 0, -1), \quad \nabla g(\mathbf{0}) = (2, 1, -1, -2).$$

Use this information in both (a) and (b) below.

(a) If the variables u, v, x, y are related by the equations

$$f(u, v, x, y) = 0, \qquad g(u, v, x, y) = 0,$$

evaluate the following quantities at the point $(u, v, x, y) = (0, 0, 0, 0)$:

$$\left(\frac{\partial u}{\partial x}\right)_y, \qquad \left(\frac{\partial u}{\partial x}\right)_v.$$

(b) Evaluate $D\mathbf{F}(\mathbf{0})$, where $\mathbf{F} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{F}(x, y) = \begin{bmatrix} f(u(x,y),\, v(x,y),\, x,\, y) \\ g(u(x,y),\, v(x,y),\, x,\, y) \end{bmatrix},$$

with $\quad u(x, y) = e^x \cos y + xe^y + y - 1 \quad$ and $\quad v(x, y) = e^y(1 - \cos x) + \sin 3x.$

[13] **4.** Find all critical points of the function

$$f(x, y) = (x^2 + 3y^2)e^{1-x^2-y^2}$$

and classify each one as a local maximizer, local minimizer, or saddle point. Does f have an absolute maximum on $\mathbb{R}^2$? If so, what is it?

[15] **5.** Consider the triangle $T = \{(x, y, z) : x > 0,\ y > 0,\ z > 0,\ x + y + z = 1\}$.

(a) Is T a closed subset of $\mathbb{R}^3$? Support your answer with a complete explanation, including the definition of a closed set.

(b) Let $f(x, y, z) = x^p y^q z^r$, where $p > 0$, $q > 0$, $r > 0$ are given. Show that f has an absolute maximum over T, but no absolute minimum over T.

(c) Find the maximum value and the maximizing point for the function f in part (b). Answer in terms of p, q, r; do not assume these are integers.

[10] **6.** Find the volume of the solid region R in (x, y, z)-space defined by

$$\sqrt{x^2 + y^2} \le z \le 12 - x^2 - y^2.$$

[15] **7.** Let R denote the solid region of (x, y, z)-space defined by

$$y \ge x^2, \qquad 0 \le z \le 1 - y.$$

(a) Let $I = \iiint_R f\, dV$. Fill in the blanks in the equations shown here:

$$I = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f\, dz\, dy\, dx = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f\, dy\, dz\, dx = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f\, dx\, dy\, dz.$$

(b) If the centroid of R lies at $(\overline{x}, \overline{y}, \overline{z})$, find $\overline{x}$ and $\overline{y}$. ($\overline{z}$ is not required.)

[15] **8.** In each part of this problem, provide a precise definition of the underlined word or phrase as part of your solution. Let

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & \text{if } (x, y) \ne (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Is f <u>continuous</u> at $(0, 0)$? Give a complete justification for your answer.

(b) Show that the <u>directional derivative</u> $D_{\hat{\mathbf{u}}} f(0, 0)$ exists for every unit vector $\hat{\mathbf{u}} = (u, v)$, and evaluate it in terms of u and v.

(c) Is f <u>differentiable</u> at $(0, 0)$? Give a complete justification for your answer.

[0] **9.** [*Bonus Question*: 5 marks possible, no partial credit.]

Certain functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, all of class C^2, satisfy the following equations at every point $(x, y) \in \mathbb{R}^2$:

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0, \quad u(x, y) = f(v(x, y)).$$

It is known that v has no critical points. Prove: For some real constants a and b,

$$u(x, y) = av(x, y) + b \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

[100] **Total Marks Available**