This examination has 8 questions on 2 pages.

The University of British Columbia
Final Examinations—December 2002

Mathematics 226
Advanced Calculus I (*Professor Loewen*)

Closed book examination. Time: $2\frac{1}{2}$ hours

Notes, books, and calculators are not allowed.
Write your answers in the booklet provided. Start each solution on a separate page.

SHOW ALL YOUR WORK!

[10] **1.** Sketch the domain of integration and evaluate:

$$I = \int_0^1 \int_x^{x^{1/3}} \sqrt{1-y^4}\, dy\, dx.$$

[10] **2.** When x, y, u, v are related by the pair of equations

$$x = u^3 + v^3, \qquad y = uv - v^2,$$

the symbol $\partial u/\partial x$ has two possible interpretations. Explain what these are, and calculate both of them at the point corresponding to $u = 1$, $v = 1$.

[12] **3.** Find the absolute maximum value of $f(x, y) = x^2y^2(5 - x - y)$ in the region where $x \geq 0$ and $y \geq 0$. Justify the "absolute maximum" assertion with care, including a complete statement of any theorem you apply. (If you cannot complete this justification, you may earn partial credit by demonstrating that you have found a local maximum.)

[12] **4.** **Background Information:** Given a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a point $\mathbf{x}_0$ in $\mathbb{R}^n$, **Newton's Method** for the approximate maximization of f involves two steps:

(1) Find $Q\colon \mathbb{R}^n \to \mathbb{R}$, the best quadratic approximation for f near the point $\mathbf{x}_0$.

(2) Find a critical point for Q and call it $\mathbf{x}_1$.

In good cases, the critical point $\mathbf{x}_1$ maximizes Q, and lies closer to a local maximizer for f than the original point $\mathbf{x}_0$. (The process can be repeated.)

Action Request: Using $f(x, y) = x^2y^2(5 - x - y)$ as in Question 3, and $\mathbf{x}_0 = (1, 1)$, apply Newton's Method as described above to find $\mathbf{x}_1$. Is this a "good case"?

[12] **5.** Find $J = \iiint_R z\, dV$, where R is the subset of $\mathbb{R}^3$ defined by

$$x \geq 0, \qquad y \geq 0, \qquad x^2 + y^2 \leq z \leq \sqrt{12 - x^2 - y^2}.$$

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[15] **6.** An ant crawls on the surface of a rugby ball: the surface obeys

$$x^2 + \frac{y^2}{2} + z^2 = 1.$$

The temperature (in °C) at each point (x, y, z) on this surface is given by

$$T(x, y, z) = \frac{8}{\sqrt{2}}\, yz \sin\left(\frac{\pi}{2}x\right).$$

As the ant passes through the point $P = (\frac{1}{2}, 1, \frac{1}{2})$, it follows a path that makes its temperature increase most rapidly.

(a) Find a vector tangent to the ant's path at P.

(b) If the ant's speed is β units/second, find its instantaneous velocity vector and its perceived rate of change of temperature at point P. Give units for your answers.

[15] **7.** Let S denote the part of the sphere $x^2+y^2+z^2 = 5r^2$ where $x > 0$, $y > 0$, and $z > 0$.

(a) Find the maximum value of $3\ln x + \ln y + \ln z$ over S.

(b) Use the result in (a) to prove that for all positive real numbers a, b, c,

$$a^3bc \le 27\left(\frac{a+b+c}{5}\right)^5.$$

[14] **8.** (a) Assuming $f\colon \mathbb{R}^2 \to \mathbb{R}$, give precise definitions for these statements:

(i) f is continuous at $(0, 0)$, and (ii) f is differentiable at $(0, 0)$.

Parts (b)–(d) refer to the specific function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{x^2(1+\pi y) + y^2}{x^2 + y^2}, & \text{if } (x, y) \ne (0, 0), \\ 1, & \text{if } (x, y) = (0, 0). \end{cases}$$

(b) Prove that f is continuous at $(0, 0)$.

(c) Find $\partial_1 f(0, 0)$ and $\partial_2 f(0, 0)$.

(d) Prove that f is not differentiable at $(0, 0)$. [Clue: Consider $f(t, t)$.]

9. BONUS QUESTION (5 MARKS):

Prove: *Every continuously differentiable function $F\colon \mathbb{R}^3 \to \mathbb{R}$ such that*

at every point (x, y, z), $\nabla F(x, y, z)$ is parallel to $(z, e^y, z\cos(x))$,

satisfies $F\left(\frac{\pi}{2}, 0, -a\right) = F\left(\frac{\pi}{2}, 0, a\right)$ *for every* $a > 0$.